Triangulating submanifolds: An elementary and quantified version of Whitney's method

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Abstract

We quantize Whitney's construction to prove the existence of a triangulation for any C^2 manifold, so that we get an algorithm with explicit bounds. We also give a new elementary proof, which is completely geometric.

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1 Introduction

The question whether every C^1 manifold admits a triangulation was of great importance to topologists in the first half of the 20th century. This question was answered in the affirmative by Cairns [11], see also Whitehead [26]. However the first proofs were complicated and not very geometric, let alone algorithmic. It was Whitney [27, Chapter IV], who eventually gave an insightful geometric constructive proof. Here, we will be reproving Theorem 12A of [27, Section IV.12], in a more quantitative/algorithmic fashion for C^2 manifolds:

▶ **Theorem 1.** *Every compact n-dimensional C^2 manifold $\mathcal{M}$ embedded in $\mathbb{R}^d$ with reach* $\mathrm{rch}(\mathcal{M})$ *admits a triangulation.*

By more quantitative, we mean that instead of being satisfied with the existence of constants that are used in the construction, we want to provide explicit bounds in terms of the reach of the manifold, which we shall assume to be positive. The reach was introduced by Federer [19], as the minimal distance between a set $\mathcal{M}$ and its medial axis. The medial axis consists of points in ambient space that do not have a unique closest point on $\mathcal{M}$. It is not too difficult to generalize the precise quantities to the setting where the manifold is $C^{1,1}$ at a small cost, see Appendix B.

Note that Theorem 1 implies that any C^1 manifolds admits a triangulation. This is because any C^1 manifold can be smoothed (see for example [20]) and Whitney's own embedding theorem (see [27, Section IV.1]) gives an embedding in $\mathbb{R}^d$.

Triangulations in computational geometry/topology are most often based on Voronoi/Delaunay triangulations of the input point set, see for example [3, 4, 9, 12, 15]. Whitney's construction is of a quite different nature. He uses an ambient triangulation and construct the triangulation of the manifold $\mathcal{M}$ based on the intersections of $\mathcal{M}$ with this triangulation. In this paper, we have chosen this ambient triangulation $\tilde{\mathcal{T}}$ to be (a perturbation of) a Coxeter triangulation $\mathcal{T}$ of type $\tilde{A}_d$. A Coxeter triangulation of type $\tilde{A}_d$ is Delaunay protected, a concept we'll recall in detail in Section 4. Delaunay protection gives that the triangulation is stable under perturbations. This property simplifies the proof, which in fact was one of the motivations for our choice. Moreover, Coxeter triangulations can be stored very compactly, in contrast with previous work [9, 12] on Delaunay triangulations.

The approach of the proof of correctness of the method, that we present in this paper, focuses on proving that after perturbing the ambient triangulation the intersection of each d-simplex in the triangulation $\tilde{\mathcal{T}}$ with $\mathcal{M}$ is a slightly deformed n-dimensional convex polytope. The triangulation K of $\mathcal{M}$ consists of a barycentric subdivision of a straightened version of these polytopes. This may remind the reader of the general result on CW-complexes, see [21], which was exploited by Edelsbrunner and Shah [18] for their triangulation result. This interpretation of Whitney's triangulation method is different from the Whitney's original proof where the homeomorphism is given by the closest point projection and uses techniques which we also exploited in [8]. Here we construct 'normals' and a tubular neighbourhood for K that is compatible with the ambient triangulation $\tilde{\mathcal{T}}$ and prove that the projection along these 'normals' is a homeomorphism. We believe that the tubular neighbourhood we construct is of independent interest. Because we have a bound on the size of the tubular neighbourhood of K and $\mathcal{M}$ lies in this neighbourhood, we automatically bound the Hausdorff distance between the two. A bound on the difference between the normals of K and $\mathcal{M}$ is also provided. Thanks to our choice of ambient triangulation and our homeomorphism proof, this entire paper is elementary in the sense that no topological results are needed, all arguments are geometrical.

In addition to the more quantitative/algorithmic approach, the purely geometrical homeomorphism proof, the link with the closed ball property, the tubular neighbourhood for the triangulation K, and a bound on the Hausdorff distance, we also give different proofs for a fair number of Whitney's intermediate results.

The algorithm described in this paper is being implemented at the moment. The results will be reported in a companion paper.

All proofs of statements that are not recalled from other papers are given in the appendix.

2 The algorithm and overview

2.1 The algorithm

The algorithm takes as input an n-dimensional C^2 manifold $\mathcal{M} \subset \mathbb{R}^d$ with reach $\text{rch}(\mathcal{M})$, and outputs the triangulation K of $\mathcal{M}$. The algorithm based on Whitney's construction consists of two parts.

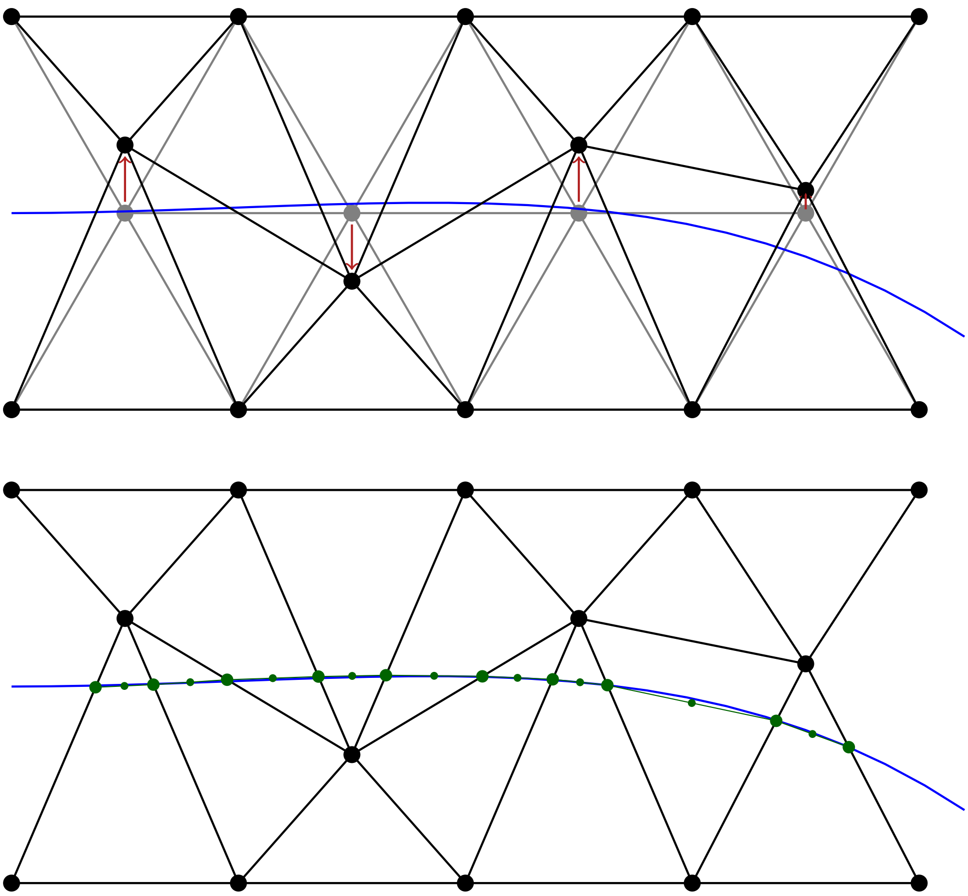

Figure 1 The two parts of the algorithm: Part 1, where we perturb the vertices of the ambient triangulation, is depicted on top. Part 2, where the triangulation is constructed from the points of intersection of $\mathcal{M}$ and the edges, is depicted below.

Part 1. This part of the algorithm handles the ambient triangulation and consists of the following steps:

- Choose a Coxeter triangulation $\mathcal{T}$ of type $\tilde{A}_d$ of $\mathbb{R}^d$ that is sufficiently fine (as determined by the longest edge length L, see (8)).
- Perturb the vertices of $\mathcal{T}$ slightly (see (14)) into a $\tilde{\mathcal{T}}$ (with the same combinatorial structure), such that all simplices in $\tilde{\mathcal{T}}$ of dimension at most $d-n-1$ are sufficiently far away from the manifold, see (11). This is done as follows: One maintains a list of vertices and simplices of $\tilde{\mathcal{T}}_i$, starting with an empty list and adding perturbed vertices, keeping the combinatorial structure of $\mathcal{T}$ intact. This means that if $\tau = \{v_{j_1}, \dots, v_{j_k}\}$ is a simplex in $\mathcal{T}$ and $\tilde{v}_{j_1}, \dots, \tilde{v}_{j_k} \in \tilde{\mathcal{T}}_i$, where $\tilde{v}_i$ denotes the perturbed vertex v_i, then $\tilde{\tau} = \{\tilde{v}_{j_1}, \dots, \tilde{v}_{j_k}\}$ is a simplex in $\tilde{\mathcal{T}}_i$. To this list, one first adds all vertices of $\mathcal{T}$ that are very far from $\mathcal{M}$, as well as the simplices with these vertices (see Case 1 of Section 5.2). For a vertex v_i that is relatively close to $\mathcal{M}$ (Case 2), one goes through the following procedure. We first pick a point $p \in \mathcal{M}$ that is not too far from v_i. We then consider all $\tau'_j \subset \tilde{\mathcal{T}}_{i-1}$ of dimension at most $d-n-2$, such that the join $v_i * \tau'_j$ lies in $\tilde{\mathcal{T}}_i$. For all such τ'_j we consider $\mathrm{span}(\tau'_j, T_p\mathcal{M})$ and we pick our perturbed v_i, that is $\tilde{v}_i$, such that it lies sufficiently far from the union of these spans (see (27)).

Note that we only require limited knowledge of the manifold. Given a vertex v_i we need to be able to find a point on $\mathcal{M}$ that is close to v_i or know if v_i is far from $\mathcal{M}$ and we need access to $T\mathcal{M}$ in a finite sufficiently dense set of points (so that for every point v_i that is close to $\mathcal{M}$ we have a linear approximation of $\mathcal{M}$). We assume we have two oracles for the two operations. There are no fundamental difficulties in including small uncertainties in our knowledge of the close points or the tangent spaces, but the analysis would be more complicated. If we can sample $\mathcal{M}$ densely finding close points is algorithmically not difficult. Methods to estimate the tangent space have been described in [1].

Complexity of part 1. The complexity of the perturbation (per vertex) of the algorithm is dominated by the number of simplices τ'_j that we have to consider. This number is bounded by the number of simplices of at most dimension $d-n-2$ in the star of a vertex in a Coxeter triangulation plus 1, see (3) below. The number of simplices in turn is bounded by $(d-n)^d d^{(d-n)}$, see claim 15. This compares favourably with the complexity of the perturbation method in [5] for Delaunay triangulations, which is of order $\mathcal{O}(2^{d^2})$. A full analysis of the complexity of the algorithm, including basic operations on Coxeter triangulations, will be reported upon in a separate paper.

Part 2. The construction of the triangulation of $\mathcal{M}$ is now straightforward barycentric subdivision, for each $\tau^k \in \tilde{\mathcal{T}}$, of dimension k that contains a part of $\mathcal{M}$, we pick a point $v(\tau^k)$ in τ^k, see (19). For any sequence $\tau^{d-n} \subset \tau^{d-n+1} \subset \cdots \subset \tau^d$, such that all simplices in the sequence contain a part of $\mathcal{M}$ we add a simplex $\{v(\tau^{d-n}), \ldots, v(\tau^d)\}$ to a simplicial complex K. If we have done this for all simplices that contain $\mathcal{M}$, K is a triangulation of $\mathcal{M}$.

For this second part we need an oracle that is able to tell us if the intersection between $\mathcal{M}$ and $\tau^{d-n} \in \tilde{\mathcal{T}}$ is non-empty and if so, gives us the point of intersection. As we'll see in Section 6.1, it would in fact suffice to be able to find intersections between tangent planes and simplices.

2.2 Outline and overview of the proof

This paper is dedicated to the correctness proof of the algorithm presented above. After some background sections dedicated to manifolds of positive reach and Coxeter triangulations and their stability under perturbations, we continue with the perturbation algorithm.

The ambient triangulation. In Section 5, we both give the details of the perturbation of the vertices.

The triangulation of $\mathcal{M}$. In Section 6, the triangulation K of $\mathcal{M}$ is defined and an important quality bound for the simplices is given.

The triangulation proof. Section 7 is dedicated to proving that K is a triangulation of $\mathcal{M}$. The proof is quite different from the approach Whitney described, which uses the closest point projection onto $\mathcal{M}$. Here we construct a tubular neighbourhood and 'normals' around the triangulation K, which is adapted to the ambient triangulation $\tilde{\mathcal{T}}$. We then prove that the projection using these 'normals' gives a piecewise smooth homeomorphism from $\tau^d \cap \mathcal{M}$ to $\tau^d \cap K$, where $\tau^d \in \tilde{\mathcal{T}}$ is d-dimensional. Because the construction is compatible on the faces of d dimensional simplices the global result immediately follows. A more detailed overview of the homeomorphism proof is given in Section 7.

3 Manifolds, tangent spaces, distances and angles

In this section, we discuss some general results that will be of use. The manifold $\mathcal{M} \subset \mathbb{R}^d$ is a compact C^2 manifold with reach $\mathrm{rch}(\mathcal{M})$.

We adhere as much as possible to the same notation as used in [10]. The tangent bundle will be denoted by $T\mathcal{M}$, while the tangent space at a point p is written as $T_p\mathcal{M}$. Similarly, $N\mathcal{M}$ is the normal bundle and $N_p\mathcal{M}$ a normal space. Distances on the manifold will be indicated by $d_{\mathcal{M}}(\cdot,\cdot)$, while we write $d(\cdot,\cdot)$ for distances in the ambient Euclidean space, and $|\cdot|$ for the length of vectors. A ball centred at x with radius r is denoted by $B(x,r)$.

The closest point projection of point x in the ambient space, such that $d(x,\mathcal{M}) < \mathrm{rch}(\mathcal{M})$, onto $\mathcal{M}$ is denoted by $\pi_{\mathcal{M}}(x)$. The orthogonal projection onto the tangent $T_p\mathcal{M}$ is denoted by $\pi_{T_p\mathcal{M}}(x)$.

We will use a result from [10], which improves upon previous works such as Niyogi, Smale and Weinberger [23]:

▶ **Lemma 2** (Lemma 6 and Corollary 3 of [10]). *Suppose that $\mathcal{M}$ is C^2 and let $p,q \in \mathcal{M}$, then*

$$\angle(T_p\mathcal{M}, T_q\mathcal{M}) \leq \frac{d_{\mathcal{M}}(p,q)}{\mathrm{rch}(\mathcal{M})} \qquad \textit{and} \qquad \sin\left(\frac{\angle(T_p\mathcal{M}, T_q\mathcal{M})}{2}\right) \leq \frac{|p-q|}{2\mathrm{rch}(\mathcal{M})}.$$

We now prove that the projection onto the tangent space is a diffeomorphism in a neighbourhood of size the reach of the manifold. This improves upon previous results by Niyogi, Smale, and Weinberger [23] in terms of the size of the neighbourhood, and is a more quantitative version of results by Whitney [27].

We first recall some notation. Similarly to [10], we let $C(T_p\mathcal{M}, r_1, r_2)$ denote the 'filled cylinder' given by all points that project orthogonally onto a ball of radius r_1 in $T_p\mathcal{M}$ and whose distance to this ball is at most r_2. We write $\mathring{C}(T_p\mathcal{M}, r_1, r_2)$ for the open cylinder. We now have:

▶ **Lemma 3.** *Suppose that $\mathcal{M}$ is C^2 and $p \in \mathcal{M}$, then for all $r < \mathrm{rch}(\mathcal{M})$, the projection $\pi_{T_p\mathcal{M}}$ onto the tangent space $T_p\mathcal{M}$, restricted to $\mathcal{M} \cap \mathring{C}(T_p\mathcal{M}, r, \mathrm{rch}(\mathcal{M}))$ is a diffeomorphism onto the open ball $B_{T_p\mathcal{M}}(r)$ of radius r in $T_p\mathcal{M}$.*

It is clear by considering the sphere that this result is tight. See Appendix B for some remarks on these results in the $C^{1,1}$ setting.

▶ **Definition 4.** We shall write π_p as an abbreviation for the restriction (of the domain) of $\pi_{T_p\mathcal{M}}$ to $\mathcal{M} \cap \mathring{C}(T_p\mathcal{M}, \mathrm{rch}(\mathcal{M}), \mathrm{rch}(\mathcal{M}))$ and π_p^{-1} for its inverse.

We now also immediately have a quantitative version of Lemma IV.8a of [27]:

▶ **Lemma 5.** *Suppose that $\mathcal{M}$ is C^2, then for all $r < \mathrm{rch}(\mathcal{M})$*

$$d(p, M \setminus C(T_p\mathcal{M}, r, \mathrm{rch}(\mathcal{M}))) = d(p, M \setminus \pi_p^{-1}(B_{T_p\mathcal{M}}(r))) \geq r. \tag{1}$$

We shall also need the following bound on the (local) distance between a tangent space and the manifold.

▶ **Lemma 6** (Distance to Manifold, Lemma 11 of [10]). *Let $\mathcal{M}$ be a manifold of positive reach. Suppose that $w \in T_p\mathcal{M}$ and $|w-p| < \mathrm{rch}(\mathcal{M})$. Let $\pi_p^{-1}(w)$ be as in Definition 4. Then*

$$|\pi_p^{-1}(w) - w| \leq \left(1 - \sqrt{1 - \left(\frac{|w-p|}{\mathrm{rch}(\mathcal{M})}\right)^2}\right) \mathrm{rch}(\mathcal{M}).$$

This is attained for the sphere of radius $\mathrm{rch}(\mathcal{M})$.

4 Coxeter triangulations, Delaunay protection and stability

Coxeter triangulations [14] of Euclidean space play a significant role in our work. They combine many of the advantages of cubes, with the advantages of triangulations. They are also attractive from the geometrical perspective, because they provide simplices with very good quality and some particular Coxeter triangulations are Delaunay protected and thus very stable Delaunay triangulations. We will now very briefly introduce both the concepts of Coxeter triangulations and Delaunay protection, but refer to [13] for more details on Coxeter triangulations and to [5, 6] for Delaunay protection.

▶ **Definition 7.** A monohedral[1] triangulation is called a *Coxeter triangulation* if all its d-simplices can be obtained by consecutive orthogonal reflections through facets of the d-simplices in the triangulation.

This definition imposes very strong constraints on the geometry of the simplices, implying that there are only a small number of such triangulations in each dimension. Most of these triangulations are part of 4 families for which there is one member for (almost) every dimension d. We will focus on one such family, $\tilde{A}_d$, which is Delaunay protected.

Protection.

▶ **Definition 8.** The *protection* of a d-simplex σ in a Delaunay triangulation on a point set P is the minimal distance of points in $P \setminus \sigma$ to the circumscribed ball of σ:

$$\delta(\sigma) = \inf_{p \in P \setminus \sigma} d(p, B(\sigma)), \text{ where } B(\sigma) \text{ is the circumscribed ball of } \sigma.$$

The *protection* δ of a Delaunay triangulation $\mathcal{T}$ is the infimum over the d-simplices of the triangulation: $\delta = \inf_{\sigma \in \mathcal{T}} \delta(\sigma)$. A triangulation with a positive protection is called *protected*.

The proof that $\tilde{A}_d$ triangulations are protected can be found in [13, Section 6]. We shall denote the triangulation of this type by $\mathcal{T}$.

Stability. In the triangulation proof below we need that a perturbation $\tilde{\mathcal{T}}$ of our initial ambient triangulation ($\mathcal{T}$ of type $\tilde{A}_d$) is still a triangulation of $\mathbb{R}^d$. Because Whitney did not use a protected Delaunay triangulation he needs a non-trivial topological argument to establish this, see [27, Appendix Section II.16]. The argument for stability of triangulations for $\tilde{A}$ type Coxeter triangulations is much simpler, because it is a Delaunay triangulation and δ-protected, see [13]. Before we can recall this result we need to introduce some notation:

- The minimal altitude or height, denoted by $\min \mathrm{alt}$, is the minimum over all vertices of the altitude, that is the distance from a vertex to the affine hull of the opposite face. $t(\tau)$ denotes the thickness of a simplex τ, that is the ratio of the minimal altitude to the maximal edge length. We write $t(\mathcal{T})$ for thickness of any simplex in $\mathcal{T}$.
- We can think of the vertices of $\mathcal{T}$ as an (ϵ, μ)-net. Here μ is the separation (for Coxeter triangulations, the shortest edge length in $\mathcal{T}$), and ϵ the sampling density (which is the circumradius of the simplices in the Coxeter triangulation). We write μ_0 for the normalized separation, that is $\mu = \mu_0 \epsilon$.
- For any complex K, $L(K)$ denotes the longest edge length in K. We use the abbreviations $L = L(\mathcal{T})$ and $\tilde{L} = L(\tilde{\mathcal{T}})$.

Theorem 4.14 of [5] immediately gives:

[1] A triangulation of $\mathbb{R}^d$ is called *monohedral* if all its d-simplices are congruent.

▶ **Corollary 9.** *The triangulation $\mathcal{T}$ is (combinatorially) stable under a $\tilde{c}L$-perturbation, meaning that $d(v_i - \tilde{v}_i) \leq \tilde{c}L$, as long as*

$$\tilde{c}L \leq \frac{t(\mathcal{T})\mu_0}{18d}\delta. \tag{2}$$

We claim the following concerning the behaviour of $\tilde{c}$

▶ **Claim 10.**

$$\tilde{c} \leq \frac{t(\sigma)\mu_0}{18d}\frac{\delta}{L} \leq \sqrt{2}\frac{\sqrt{d^2+2d+24}-\sqrt{d^2+2d}}{9d^{3/2}\sqrt{d+2}(d+1)} \sim \frac{\sqrt{32}}{3d^4},$$

Thickness and angles. The quality of simplices and the control over the alignment of the simplices with the manifold is an essential part of the triangulation proof, for which we need two basic results. Similar statements can be found in Section IV.14 and Section IV.15 of [27].

We remind ourselves of the following:

▶ **Lemma 11** (Thickness under distortion [17, Lemma 7]). *Suppose that $\sigma = \{v_0, \dots, v_k\}$ and $\tilde{\sigma} = \{\tilde{v}_0, \dots, \tilde{v}_k\}$ are two k-simplices in $\mathbb{R}^d$ such that $||v_i - v_j| - |\tilde{v}_i - \tilde{v}_j|| \leq c_0 L(\sigma)$ for all $0 \leq i < j \leq k$. If $c_0 \leq \frac{t(\sigma)^2}{4}$, then $t(\tilde{\sigma}) \geq \frac{4}{5\sqrt{k}}(1 - \frac{4c_0}{t(\sigma)^2})t(\sigma)$.*

We can now state a variation of Whitney's alignment result, see [27, Section IV.15].

▶ **Lemma 12** (Whitney's angle bound). *Suppose σ is a j-simplex of $\mathbb{R}^d$, $j < d$, whose vertices all lie within a distance d_{max} from a k-dimensional affine space $A_0 \subset \mathbb{R}^d$ with $k \geq j$. Then*

$$\sin \angle(\operatorname{aff}(\sigma), A_0) \leq \frac{(j+1)\, d_{max}}{\min \operatorname{alt}(\sigma)}.$$

Simplices in a star in a triangulation of type $\tilde{\mathbf{A}}_\mathbf{d}$. The precise number of simplices in the star of a vertex plays an important role in the volume estimates in Section 5. We will now give an explicit bound on this number.

In general the $(d-k)$-faces of a Voronoi cell correspond to the k-faces in the Delaunay dual. The triangulation $\mathcal{T}$ is Delaunay and the dual of a vertex is a permutahedron, see [13]. We now remind ourselves of the following definition, see [2], and lemma:

▶ **Definition 13.** Let $S(d,k)$ be the Stirling number of the second kind, which is the number of ways to partition a set of d elements into k non-empty subsets, that is

$$S(d,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{i}(k-j)^d.$$

▶ **Corollary 14** (Corollary 3.15 of [22]). *The number of $(d+1-k)$-faces of the permutahedron is $k!S(d+1,k)$.*

By duality, the lemma immediately gives us the number N_k of k-faces that contain a given vertex in $\mathcal{T}$, $N_k = k!S(d+1,k)$. We also write

$$N_{\leq k} = 2 + \sum_{j=1}^{k} j!S(d+1,j), \tag{3}$$

which is an upper bound on the total number of faces of dimension less or equal to k that contain a given vertex. We have added 2 because we want to have a safety margin if we have to consider the empty set (as will be apparent in (16)), and have a strict inequality. We now claim the following:

▶ **Claim 15.** *We have $N_{\leq k} \lesssim k^d d^k$.*

5 Perturbing the ambient triangulation

This section is dedicated to the perturbation of the Coxeter triangulation such that the manifold is sufficiently far from the simplices of dimension at most $d-n-1$ in $\tilde{\mathcal{T}}$.

- In Section 5.1, we prove that it is possible to perturb the points as described in the second step of part 1 of the algorithm. This involves a significant amount of volume estimates, which are now completely quantized. Indicate how fine the ambient triangulation $\mathcal{T}$ has to be compared to $\mathrm{rch}(\mathcal{M})$
- In Section 5.2, we define the perturbation and prove that this in fact gives a triangulation for which the low dimensional simplices lie sufficiently far from the manifold.

The proofs of the results in Section 5.2 rely on Appendix A. We shall indicate the corresponding sections in Whitney [27] between brackets in the title of the section, when appropriate.

5.1 The complex $\tilde{\mathcal{T}}$ (Section IV.18 of [27])

Before we can dive into the algorithmic construction of the perturbed complex $\tilde{\mathcal{T}}$, we need to fix some constants and give some explicit bounds on the constants.

Balls and exclusion volumes. Let $B^d(r)$ be any ball in $\mathbb{R}^d$ of radius r. We now define $\bar{\rho}_1 > 0$ as follows: For any two parallel $(d-1)$-hyperplanes whose distance apart is less than $2\bar{\rho}_1 r$, the intersection of the slab between the two hyperplanes with the ball $B^d(r)$ is denoted by $\mathcal{S}$. Now, $\bar{\rho}_1$ is the largest number such that the volume (vol) of any $\mathcal{S}$ satisfies

$$\mathrm{vol}(\mathcal{S}) \leq \frac{\mathrm{vol}(B^d(r))}{2N_{\leq d-n-1}},$$

with $N_{\leq d-n}$ as in (3). A precise bound on $\bar{\rho}_1$ can be given, see Remark 35 in Appendix C. We will use an easier bound ρ_1, at the cost of weakening the result:

▶ **Lemma 16.** *We have*

$$\bar{\rho}_1 \geq \rho_1 = \begin{cases} \frac{2^{2k-2}(k!)^2}{\pi(2k)!N_{\leq d-n-1}} & \text{if } d = 2k \\ \frac{(2k)!}{2^{2k+2}k!(k-1)!N_{\leq d-n-1}} & \text{if } d = 2k-1. \end{cases} \tag{4}$$

Note that

$$\rho_1 \sim \frac{1}{\sqrt{d}N_{\leq d-n-1}}.$$

The coarseness of $\mathcal{T}$. As mentioned, we perturb the vertices of a Coxeter triangulation. The maximal distance between an unperturbed vertex v_i and the associated perturbed vertex $\tilde{v}_i$ is $\tilde{c}L$. We will assume that $\tilde{c}$ satisfies the bounds below in (14).

We are now ready to introduce the demands on the triangulation of ambient space. We start by bounding the scale of the Coxeter triangulation $\mathcal{T}$ by bounding the longest edge length. We do this by giving some constants. We define α_1 and α_k by a recursion relation as follows

$$\alpha_1 = \frac{4}{3}\rho_1\tilde{c} \qquad\qquad \frac{2}{3}\alpha_{k-1}\tilde{c}\rho_1 = \alpha_k, \tag{5}$$

that is $\alpha_k = \frac{2^{k+1}}{3^k}\rho_1^k \tilde{c}^k$. These definitions play an essential role in the volume estimates for the perturbation of the vertices, that are necessary to guarantee quality. Note that α_k is extremely small. In particular, we shall have that

$$\alpha_k \leq \frac{1}{18^k}, \tag{6}$$

because of the way that $\tilde{c}$ will be chosen, see (15), ρ_1 is also very small. Furthermore we notice that $\alpha_k < \alpha_{k-1}$. To make sure the formulae do not become too big, we introduce the notation

$$\zeta = \frac{8}{15\sqrt{d}\binom{d}{d-n}\cdot(1+2\,\tilde{c})}\left(1 - \frac{8\tilde{c}}{t(\mathcal{T})^2}\right)t(\mathcal{T}). \tag{7}$$

Note that ζ depends on both the ambient and intrinsic dimension and the perturbation parameter $\tilde{c}$. We set the coarseness of the ambient triangulation by demanding that L satisfies

$$\left(1 - \sqrt{1 - \left(\frac{6L(\mathcal{T})}{\mathrm{rch}(\mathcal{M})}\right)^2}\right)\mathrm{rch}(\mathcal{M}) = \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}L, \tag{8}$$

or equivalently

$$\frac{L}{\mathrm{rch}(\mathcal{M})} = \frac{2\frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}}{\left(\frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}\right)^2 + 6^2}. \tag{9}$$

Note that

$$\frac{L}{\mathrm{rch}(\mathcal{M})} < \frac{(\alpha_{d-n})^{4+2n}}{54(n+1)^2}\zeta^n < \frac{(\alpha_{d-n})^2}{54}, \qquad \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n} < \frac{(\alpha_{d-n})^2}{3} \leq \frac{\alpha_{d-n}}{3}, \tag{10}$$

which will be often used below to simplify expressions.

▶ Remark. We have to choose the right hand side in (8) very small, because the bounds on the quality of the simplices that will make up the triangulations are very weak. The details of these estimates can be found in Lemma 26.

$(\mathbf{d-n-1})$-skeleton safe triangulations. We shall denote the simplices by τ and σ. We will use lower indices to distinguish simplices, while upper indices will stress the dimension, for example τ_j^k is a simplex of dimension k.

▶ **Definition 17** ($(\mathbf{d-n-1})$-skeleton safe triangulations)**.** We say that a perturbated triangulation $\tilde{\mathcal{T}}$ of $\mathcal{T}$ in $\mathbb{R}^d$ is $(d-n-1)$-*skeleton safe* with respect to the n-dimensional manifold $\mathcal{M}$ if

$$d(\tau^k, \mathcal{M}) > \alpha_k L, \tag{11}$$

for all faces τ^k in $\tilde{\mathcal{T}}$, with $k \leq d-n-1$, and

$$\tilde{L} < \frac{13}{12}L \tag{12}$$

$$t(\tilde{\mathcal{T}}) \geq \frac{4}{5\sqrt{d}}\left(1 - \frac{8\tilde{c}}{t(\mathcal{T})^2}\right)t(\mathcal{T}). \tag{13}$$

5.2 Perturbing the vertices (Section IV.18)

We now discuss the details of the perturbation scheme that we described in the algorithm section. The perturbation scheme follows Whitney and is inductive.

Construction of $\tilde{\mathcal{T}}$. Let $v_1, v_2, \dots$ be the vertices of $\mathcal{T}$, we are going to inductively choose new vertices $\tilde{v}_1, \tilde{v}_2, \dots$ for $\tilde{\mathcal{T}}$, with

$$|v_i - \tilde{v}_i| \leq \tilde{c}L = \min\left\{\frac{t(\mathcal{T})\mu_0}{18d}\delta, \frac{1}{24}t(\mathcal{T})^2 L\right\}, \tag{14}$$

using the notation of Section 4. Notice that because $t(\mathcal{T}) \leq 1$, by definition of the thickness of a simplex, we have

$$\tilde{c} \leq \frac{1}{24}. \tag{15}$$

With this bound we immediately have that (12) is satisfied. We also claim the following:

▶ **Claim 18.** *$\tilde{\mathcal{T}}$ has the same combinatorial structure as $\mathcal{T}$. Moreover,* (13) *is satisfied.*

We now give the scheme where the vertices are perturbed inductively. Suppose that the vertices $\tilde{v}_1, \dots, \tilde{v}_{i-1}$ have been determined, and thus the complex $\tilde{\mathcal{T}}_{i-1}$ with these vertices. A simplex $\{\tilde{v}_{j_1} \dots \tilde{v}_{j_k}\}$ lies in $\tilde{\mathcal{T}}_{i-1}$ if and only if $\{v_{j_1} \dots v_{j_k}\}$ lies in $\mathcal{T}$. We shall now find $\tilde{v}_i$ and thus $\tilde{\mathcal{T}}_i$ so that for any $\tau^k \in \tilde{\mathcal{T}}_i$ of dimension $k \leq d-n-1$, (11) is satisfied. We distinguish two cases:

Case 1. The distance $d(v_i, \mathcal{M})$ is greater than $\frac{3}{2}L$. In this case we choose $\tilde{v}_i = v_i$. The inequality (11) now follows, because $\tilde{L} < (1+2\tilde{c})L$ and thus by the triangle inequality any simplex in $\tilde{\mathcal{T}}$ with vertex v_i is at least distance $(\frac{1}{2} - 2\tilde{c})L > \frac{5}{12}L$ from the manifold, which is much larger than $\alpha_k L < \frac{1}{18^k}L$.

Case 2. The distance $d(v_i, \mathcal{M})$ is smaller than $\frac{3}{2}L$. Let p be a point in $\mathcal{M}$ such that $d(v_i, p) < \frac{3}{2}L$. Let

$$\tau'_0 (= \emptyset), \tau'_1, \dots, \tau'_\nu \tag{16}$$

be the simplices of $\tilde{\mathcal{T}}_{i-1}$ such that the joins $\tau_j = \tau'_j * \tilde{v}_i$ are simplices of $\tilde{\mathcal{T}}$, and $\dim(\tau'_j * \tilde{v}_i) \leq d-n-1$ (and thus $\dim(\tau'_j) \leq d-n-2$), with $0 \leq j \leq \nu$. We note that $\nu \leq N_{\leq d-n-1}$, with $N_{\leq k}$ as defined in (3). We now consider the span, denoted by $\mathrm{span}(\tau'_j, T_p\mathcal{M})$, for all $0 \leq j \leq \nu$. Note that the dimension of $\mathrm{span}(\tau'_j, T_p\mathcal{M})$ is at most $(d-n-2)+n+1 = d-1$.

We now claim the following:

▶ **Claim 19.** *We can pick $\tilde{v}_i$ such that it lies sufficiently far from each span$(\tau'_j, T_p\mathcal{M})$, that is*

$$d(\tilde{v}_i, \mathit{span}(\tau'_j, T_p\mathcal{M})) \geq \rho_1 \tilde{c}L, \tag{17}$$

while it is not too far from v_i, that is $|\tilde{v}_i - v_i| \leq \tilde{c}L$.

The following claim completes Case 2 and establishes that the triangulation is $(d-n-1)$-skeleton safe:

▶ **Claim 20.** (11) *is satisfied.*

We emphasize that in the perturbation of the points it suffices to look at the tangent spaces at specific points, making this constructive proof an algorithm.

6 Constructing the triangulation of $\mathcal{M}$

Section 6.1 gives geometric consequences of the perturbation we discussed in the previous section. Most importantly we shall see that the intersections of simplices in $\tilde{\mathcal{T}}$ with $\mathcal{M}$ and those of simplices in $\tilde{\mathcal{T}}$ with the tangent space of $\mathcal{M}$ at a nearby point are equivalent. Here we again rely on Appendix A. The triangulation K of $\mathcal{M}$ is defined in Section 6.2.

6.1 The geometry of the intersection of simplices in $\tilde{\mathcal{T}}$ and $\mathcal{M}$

In this section, we discuss the geometry of simplices in $\tilde{\mathcal{T}}$ in relation to $\mathcal{M}$. We follow [27, Section IV.19], with the usual exceptions of the use of Coxeter triangulations, the thickness, and the reach to quantify the results. The proofs below also differ in a fair number of places from the original.

We first establish a lower bound on the distance between $T_p\mathcal{M}$ and simplices in the $(d-n-1)$-skeleton of $\mathcal{T}^{\mathcal{P}}$ that are close to p.

▶ **Lemma 21.** *Let $p \in \mathcal{M}$ and suppose that $\tau^k \in \tilde{\mathcal{T}}$, with $k \leq d-n-1$, such that $\tau^k \subset B(p, 6L)$, then*

$$d(\tau^k, T_p\mathcal{M}) > \frac{2}{3}\alpha_k L.$$

We can now examine the relation between intersections with the manifold and nearby tangent spaces.

▶ **Lemma 22.** *Suppose that $\mathcal{M}$ intersects $\tau^k \in \tilde{\mathcal{T}}$. Let $p \in \mathcal{M}$, such that $\tau^k \subset B(p, 6(L)$, then $T_p\mathcal{M}$ intersects τ^k.*

We can now bound the angle between simplices and tangent spaces.

▶ **Lemma 23.** *Suppose that $\mathcal{M}$ intersects $\tau^k \in \tilde{\mathcal{T}}$ and τ^k has dimension $d-n$, that is $k = d-n$. Let $p \in \mathcal{M}$, such that $\tau^k \subset B(p, 6L)$, then*

$$\sin \angle(\mathrm{aff}(\tau^k), T_p\mathcal{M}) \geq \frac{2d(T_p\mathcal{M}, \partial\tau^k)}{L + 2\tilde{c}L} \geq \frac{\frac{4}{3}\alpha_k L}{L + 2\tilde{c}L} \geq \frac{16}{10}\alpha_k.$$

Below we investigate the relation between intersections of tangent spaces and simplices and intersections between the manifold and simplices. We combine two statements of Section IV.19 in the following lemma. The proof is quite different from the original.

▶ **Lemma 24.** *If $p \in \mathcal{M}$, $\tau^k \in \tilde{\mathcal{T}}$ and $\tau^k \subset B(p, 6L)$, and moreover $T_p\mathcal{M}$ intersects τ^k, then $k \geq d-n$ and $\mathcal{M}$ intersects τ^k. If $k = d-n$ this point is unique, which in particular means that every simplex of dimension $d-n$ contains at most one point of $\mathcal{M}$*

Finally, we study the faces of a simplex that intersects $\mathcal{M}$. This is essential for the barycentric subdivision in part 2 of the algorithm. The proof is identical to the original.

▶ **Lemma 25.** *If $\mathcal{M}$ intersects $\tau = \{v_0, \ldots, v_r\} \in \tilde{\mathcal{T}}$, then for each $v_i \in \tau$, there exists some $(d-n)$-face τ' of τ such that $v_i \in \tau'$ and τ' intersects $\mathcal{M}$.*

6.2 The triangulation of $\mathcal{M}$: The complex K (Section IV.20 of [27])

The construction of the complex follows Section IV.20 of [27].

In each simplex τ of $\tilde{\mathcal{T}}$ that intersects $\mathcal{M}$ we choose a point $v(\tau)$ and construct a complex K with these points as vertices. The construction goes via barycentric subdivision. For each sequence $\tau_0 \subset \tau_1 \subset \cdots \subset \tau_k$ of distinct simplices in $\tilde{\mathcal{T}}$ such that τ_0 intersects $\mathcal{M}$,

$$\sigma^k = \{v(\tau_0), \ldots, v(\tau_k)\} \tag{18}$$

will be a simplex of K. The definition of $v(\tau)$ depends on the dimension of τ:

- If τ is a simplex of dimension $d-n$, then there is an unique point of intersection with $\mathcal{M}$, due to Lemma 24. We define $v(\tau)$ to be this unique point.
- If τ has dimension greater than $d-n$, then we consider the faces $\tau_1^{d-n}, \ldots, \tau_j^{d-n}$ of τ of dimension $d-n$ that intersect $\mathcal{M}$. These faces exist thanks to Lemma 25. We now define $v(\tau)$ as follows:

$$v(\tau) = \frac{1}{j}\left(v(\tau_1^{d-n}) + \cdots + v(\tau_j^{d-n})\right). \tag{19}$$

▶ Remark. We stress that thanks to Lemma 24, choosing the point $v(\tau^{d-n})$ to be the point of intersection with $T_p\mathcal{M}$, assuming p is sufficiently close, locally gives the same combinatorial structure as intersections with $\mathcal{M}$. We also stress that for the combinatorial structure it does not really matter where $\mathcal{M}$ intersects a simplex of $\tilde{\mathcal{T}}$, as long as it does.

We can now prove the following bound on the altitudes of the simplices we constructed in this manner.

▶ **Lemma 26.** *Let σ^n be a top dimensional simplex as defined in* (18), *then*

$$\min \operatorname{alt}(\sigma^n) > \zeta^n (\alpha_{d-n-1})^n \tilde{L},$$

where $\min \operatorname{alt}$ *denotes the minimal altitude or height, and we used the notation ζ, as defined in* (7).

7 The triangulation proof

Given the triangulation $\tilde{\mathcal{T}}$, we want to prove that the intersection of $\mathcal{M} \cap \tau^d$ is homeomorphic to the triangulated polytope described in Section 6.2. This immediately gives a global homeomorphism between the triangulation and the manifold.

The homeomorphism we discuss in this section differs greatly from Whitney's own approach. Firstly, he used the closest point projection as a map (which does not respect simplices, meaning that the point in K and its projection may lie in different simplices of $\tilde{\mathcal{T}}$. Secondly, to prove that this map is a homeomorphism, he uses what has become known as Whitney's lemma in much the same way as in [8].

The great advantage of our approach to the homeomorphism proof is that it is extremely explicit and it is elementary in the sense that it does not rely on topological results. We also need precise bounds on the angles, which do not require deep theory, but are quite intricate.

Because we work with an ambient triangulation of type $\tilde{A}$ and we do not perturb too much, the simplices of $\tilde{\mathcal{T}}$ are Delaunay. The homeomorphism from $\mathcal{M} \cap \tau^d$ to the triangulated polytope $K \cap \tau^d$, with K as defined in Section 6.2 and $\tau^d \in \tilde{\mathcal{T}}$, gives that the intersection of any simplex in $\tilde{\mathcal{T}}$ with $\mathcal{M}$ is a topological ball of the appropriate dimension. This may remind the reader of the closed ball property of Edelsbrunner and Shah [18].

Overview homeomorphism proof. The homeomorphism proof consists of three steps:

- For each maximal simplex $\tau \in \tilde{\mathcal{T}}$ we provide a 'tubular neighbourhood' for $K \cap \tau$ adapted to τ. By this we mean that, for each point $\bar{p}$ in $K \cap \tau$, we designate a 'normal' space $\mathcal{N}_{\bar{p}}$ that has dimension equal to the codimension of $\mathcal{M}$ and K and is transversal to $K \cap \tau$. Moreover, these directions shall be chosen in a sufficiently controlled and smooth way, so that every point x in τ that is sufficiently close to K has a unique point $\bar{p}$ on $K \cap \tau$ such that $x - \bar{p} \in \mathcal{N}_{\bar{p}}$.
- We give conditions that enforce that the 'normal' spaces $\mathcal{N}_{\bar{p}}$ intersect $\mathcal{M}$ transversely. More precisely, we prove that the angle between $\tilde{N}_{\bar{p}}$ and $N_q\mathcal{M}$, for any $q \in \mathcal{M} \cap \tau$ is upper bounded by a quantity strictly less than 90 degrees.
- We conclude that the projection along $\mathcal{N}_{\bar{p}}$ gives a homeomorphism from $\mathcal{M}$ to K.

7.1 Constructing the tubular neighbourhood

We now give the construction of the 'tubular neighbourhood' of K. We use two results from the previous sections:

- The normal space is almost constant, see Lemma 2, near a simplex $\tau \in \tilde{\mathcal{T}}$, because it is small. So $T\mathcal{M}$ and $N\mathcal{M}$ near p are well approximated by $T_p\mathcal{M}$ and $N_p\mathcal{M}$.
- The angles between the normal space and those faces τ_k^{d-n} of dimension $d-n$ that intersect $\mathcal{M}$ are bounded from below by Lemma 23.

This means that the orthogonal projection map $\pi_{\mathrm{aff}(\tau_k^{d-n}) \to N_p\mathcal{M}} = \pi_{\tau_k^{d-n}}$ from the affine hull $\mathrm{aff}(\tau_k^{d-n})$ to $N_p\mathcal{M}$ is a (linear) bijection. Now note that the orthonormal basis $e_1, \dots, e_{d-n}$ of $N_p\mathcal{M}$ induces a (generally not orthonormal) basis of $\mathrm{aff}(\tau_k^{d-n})$ under the inverse image of the map $\pi_{\tau_k^{d-n}}$, denoted by $\{\pi^{-1}_{\tau_k^{d-n}}(e_j) \mid j = 1, \dots, d-n\}$.

We can now define the normal spaces for the complex K. We first do this for the vertices $v(\tau^{d-n})$ (these vertices lie on $\mathcal{M}$), secondly for general vertices of K (these vertices do not necessarily lie on $\mathcal{M}$) and finally using barycentric coordinates for arbitrary points in K.

Having defined a basis of $\mathrm{aff}(\tau_k^{d-n})$ for the vertices $v(\tau_k^{d-n})$, as defined in Section 6.2, we can construct a basis of a $(d-n)$-dimensional affine space, through each $v(\tau)$. The construction is similar to that of Section 6.2. If τ has dimension greater than $d-n$, then we consider the faces $\tau_1^{d-n}, \dots, \tau_j^{d-n}$ of τ of dimension $d-n$ that intersect $\mathcal{M}$. For each $e_i \in N_p\mathcal{M}$, we write

$$N_{v(\tau)}(e_i) = \frac{1}{j}\left(\pi^{-1}_{\tau_1^{d-n}}(e_i) + \dots + \pi^{-1}_{\tau_j^{d-n}}(e_i)\right). \tag{20}$$

Using these vectors we can define the space $\mathcal{N}_{v(\tau)} = \mathrm{span}(N_{v(\tau)}(e_i))$.

Let $\sigma^n = \{v(\tau_0^{d-n}), \dots, v(\tau_n^d)\}$ be a simplex of K. For any point $\bar{p}$ in σ^n, with barycentric coordinates $\lambda = (\lambda_0, \dots, \lambda_n)$, we define

$$N_{\bar{p}}(e_i) = \lambda_0 N_{v(\tau_0^{d-n})}(e_i) + \dots + \lambda_n N_{v(\tau_n^{d-n})}(e_i). \tag{21}$$

By defining $\mathcal{N}_{\bar{p}} = \mathrm{span}(N_{\bar{p}}(e_i))$, we get affine spaces for each point in each $\sigma^n \in K$.

▶ Remark. By construction, these spaces are consistent on the faces of simplices in K as well as with the boundaries of the d dimensional simplices in $\tilde{\mathcal{T}}$.

7.2 The size of the tubular neighbourhoods and the homeomorphism

In this section, we establish the size of the neighbourhood of K as defined by $\mathcal{N}_{\bar{p}}$.

The following angle estimate is an essential part of the estimate of the size of the neighbourhood of the triangulation K.

▶ **Lemma 27.** *Suppose that $p \in \mathcal{M}$, $\tau^d \subset B(p, 6L)$ and $\sigma^n \in K$ such that $\sigma^n \subset \tau^d$, where we regard σ^n and τ^d as subsets of $\mathbb{R}^d$. Then the angle between $T_p\mathcal{M}$ and $\mathrm{aff}(\sigma^n)$ is bounded as follows:*

$$\sin \angle(\mathrm{aff}(\sigma^n), T_p\mathcal{M}) \leq \frac{(\alpha_{d-n})^{4+n}}{6(n+1)}\zeta^n.$$

With this we can give a bound on the size of the neighbourhood of K.

▶ **Lemma 28.** *Let $\bar{p}, \bar{q} \in \sigma^n$, with barycentric coordinates $\lambda = (\lambda_0, \ldots, \lambda_n)$, $\lambda' = (\lambda'_0, \ldots, \lambda'_n)$ respectively. Suppose that $\mathcal{N}_{\bar{p}}$ and $\mathcal{N}_{\bar{q}}$ be as defined in Section 7.1. Suppose now that the intersection between $\bar{p}+\mathcal{N}_{\bar{p}}$ and $\bar{q}+\mathcal{N}_{\bar{q}}$ is non-empty. Here $\bar{p}+\mathcal{N}_{\bar{p}}$ and $\bar{q}+\mathcal{N}_{\bar{q}}$ denote the affine spaces that go through $\bar{p}$, $\bar{q}$ and are parallel to $\mathcal{N}_{\bar{p}}$, $\mathcal{N}_{\bar{q}}$, respectively. If $x \in \bar{p}+\mathcal{N}_{\bar{p}} \cap \bar{q}+\mathcal{N}_{\bar{q}}$, then*

$$d(x, \mathrm{aff}(\sigma^n)) \geq \frac{\left(\frac{16}{10}\right)^2 (\alpha_{d-n})^4}{n+1}\zeta^n(\alpha_{d-n-1})^n\tilde{L}.$$

Because, by construction, the $\mathcal{N}_p$ agree on the faces of the n-dimensional simplices in K, this provides a tubular neighbourhood for K of this size.

▶ **Lemma 29.** *Suppose $\tau^d \in \tilde{\mathcal{T}}$ and that $\mathcal{M} \cap \tau^d \neq \emptyset$. Then, $\mathcal{M} \cap \tau^d$ lies in the tubular neighbourhood of $K \cap \tau^d$ as defined in Section 7.1.*

Having established that $\mathcal{M}$ lies in the tubular neighbourhood around K we can sensibly speak about the projection from $\mathcal{M}$ to K along the direction N. Because we also have that the projection from $\mathcal{M}$ to K in the direction $\mathcal{N}$ (as defined in Section 7.1) is transversal (Because $\pi/2$ minus the angle between $\mathcal{N}_{\bar{p}}$ and $N_p\mathcal{M}$, see (33), is much bigger than the variation of the tangent/normal space as bounded by Lemma 2 and (9)) we see that $\mathcal{M} \cap \tau^d$ is homeomorphic to $K \cap \tau^d$. By construction the projection map is compatible on the boundaries of τ^d, so we also immediately have an explicit homeomorphism between $\mathcal{M}$ and K. This completes the proof of Theorem 1. We emphasize that along the way we have also given a bounds on

- the Hausdorff distance between $\mathcal{M}$ and K (Lemmas 29 and 28)
- the quality (Lemma 26)
- the variation of the tangent spaces ((33), Lemma 2 and (9) as mentioned above).

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A Some properties of affine spaces

In this appendix, we discuss two variants of lemmas from Appendix Section II.14 of [27], that are essential in the building of the triangulation, see Section 6.1 in particular. Both lemmas are due to Whitney. However, in both cases, the statement is different, because we prefer to work directly with angles and use the thickness as our quality measure. In the first case, the proof we provide differs significantly from the original. The first lemma will allow us to prove that if $T_p\mathcal{M}$ intersects a simplex $\tau \in \mathcal{T}$ and p and τ are not too far from each other then $\mathcal{M}$ intersects τ and vice versa. The second result is essential in proving that the perturbation of the vertices as described in Section 2.1 part 1, gives a triangulation for which the low dimensional simplices are sufficiently far away from the manifold.

We start with a variation on Lemma 14a of Appendix Section II.14 of [27].

▶ **Lemma 30.** *Let σ be an s-simplex and A_0 an affine n-dimensional subspace in $\mathbb{R}^d$. Assume that $s + n \geq d$ and*

$$d(A_0, \sigma) < d(A_0, \partial\sigma).$$

Then $s + n = d$, A intersects σ in a single point and

$$\sin \angle \mathrm{aff}(\sigma) A_0 \geq 2d(A_0, \partial\sigma)/L(\sigma).$$

Proof of Lemma 30. Choose $p \in \sigma$ and $q \in A_0$ such that

$$|p - q| = d(A_0, \sigma).$$

Now suppose that there is a vector $v \neq 0$ that lies in the intersection of $\mathrm{aff}(\sigma)$ and A_0. Then there exists some $c \in \mathbb{R}$ such that $p + cv \in \partial\sigma$. Because v lies in the intersection of $\mathrm{aff}(\sigma)$ and A_0, we have that $q + cv \in A_0$. Clearly translation leaves distances invariant, so

$$d(A_0, \sigma) = |p - q| = |(p + cv) - (q + cv)| \geq d(A_0, \partial\sigma),$$

which clearly contradicts the assumption. This means we can conclude that there is no such v and therefore $s + n = d$.

Because there is no v in the intersection of $\mathrm{aff}(\sigma)$ and A_0, there is a unique point $\bar{p}$ in this intersection. We'll now show that $\bar{p} \in \sigma$. I'll assume that $\bar{p} \notin \sigma$. This means in particular that $q \neq \bar{p}$. Because $d(A_0, \sigma) < d(A_0, \partial\sigma)$, $p - q$ is normal to $\mathrm{aff}(\sigma)$ and $p \in \sigma \setminus \partial\sigma$. Now consider the line from q to $\bar{p}$, which lies in A_0. The distance from a point on this line to σ decreases (at least at first) as you go from q toward $\bar{p}$. This contradicts the definition of q. We conclude that $\bar{p} \in \sigma$.

Now suppose that l_0 is a line in A_0 that goes though $\bar{p}$. In order to derive a contradiction, we assume that

$$\sin\phi < 2d(A_0, \partial\sigma)/L(\sigma),$$

where $\sin\phi$ denotes the angle between l_0 and $\mathrm{aff}(\sigma)$. Denote by $\pi_{\mathrm{aff}(\sigma)}(l_0)$ the orthogonal/closest point projection on $\mathrm{aff}(\sigma)$ of l_0. Because $\bar{p} \in \sigma$ $\pi_{\mathrm{aff}(\sigma)}(l_0)$ intersects $\partial\sigma$ at a point $\bar{q}$ and we may assume that $|\bar{p} - \bar{q}| \leq \frac{1}{2}L(\sigma)$ so that l_0 contains a point a distance

$$\frac{1}{2}L(\sigma)\sin\phi < \frac{1}{2}L(\sigma)2d(A_0, \partial\sigma)/L(\sigma) = d(A_0, \partial\sigma)$$

from $\partial\sigma$, a contradiction. Because l_0 was an arbitrary line in A_0 the result now follows. ◀

The following is a variation on Lemma 14b of Appendix Section II.14 of [27]. The proof presented here is almost identical to the original.

▶ **Lemma 31.** *Let A_1 and A_2 be two affine subspaces in $\mathbb{R}^d$, with $A_1 \subset A_2$. Let τ be a simplex in A_2, and let v be a point in $\mathbb{R}^d \setminus \tau$. Define J to be the join of τ and v. Then*

$$d(J, A_1) \geq \frac{d(\tau, A_1)d(v, A_2)}{L(J)}, \tag{22}$$

where the distances between sets are defined as $d(B, C) = \inf_{x\in B, y\in C} |x-y|$ and $L(J)$ denotes the longest edge length of an edge in J.

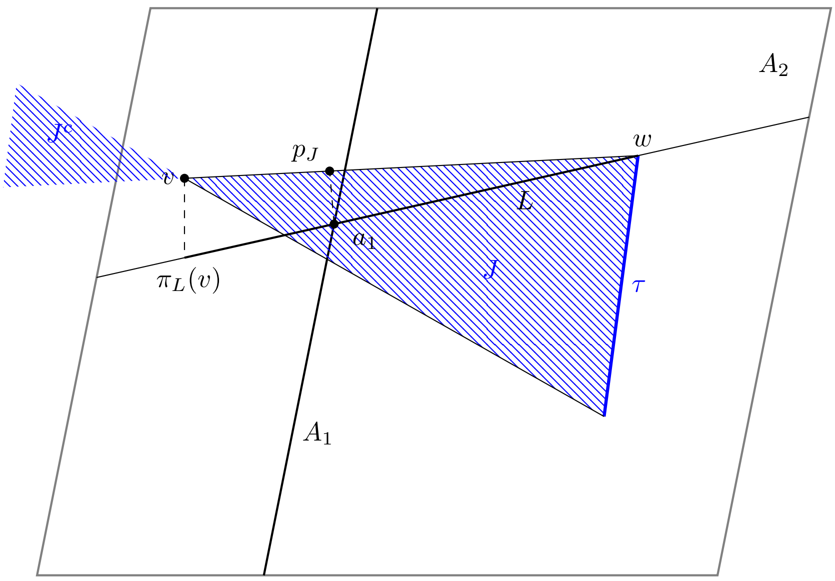

Figure 2 Notation for the proof of Lemma 31.

Proof of Lemma 31. Let us suppose that (22) is false. Let J^c be the truncated cone that consists of all half-lines that start at a point of τ and pass through v. Then we may choose $p_J \in J^c$ and $a_1 \in A_1$ so that

$$|p_J - a_1| = d(J^c, A_1),$$

by the definition of J^c and the hypothesis we also have

$$d(J^c, A_1) \leq d(J, A_1) < \frac{d(\tau, A_1)d(v, A_2)}{L(J)}. \tag{23}$$

Now suppose that p_J lies on the half line that starts at $w \in \tau$ and goes through v. Because $\tau \subset A_2$, we see that $d(v, A_2) \leq L_e(J)$. This means that (23) gives that $d(J^c, A_1) < d(\tau, A_1)$, so that $p_J \neq w$. We now immediately see that the line segment $a_1 p_J$ is orthogonal to the line that goes through w and v, which extends the half line we mentioned above. Let ℓ now be the line that goes through a_1 and w, and $\pi_\ell(v) \in \ell$ the point that is closest to v. It follows that $\pi_\ell(v)w$ is perpendicular to ℓ. Because a_1 is nearer to p_J than w, a_1 and $\pi_\ell(v)$ are on the same side of w in ℓ. This means because two of the angles are the same (and thus the third), that the triangles $p_J w a_1$ and $\pi_\ell(v)wv$ are similar. We now have that

$$d(J^c, A_1) = |p_J - a_1| = \frac{|a_1 - w||v - \pi_\ell(v)|}{|v - w|} \geq \frac{d(\tau, A_1)d(v, A_2)}{L(J)},$$

contradicting the hypothesis and thus proving the lemma. ◀

B Remark on the $C^{1,1}$ case

We now first discuss a simpler version of 3. The result in this case is weaker, but can be easily extended to the $C^{1,1}$ setting as we shall see below.

The following consequence of Lemma 2 of is a stronger version of Lemma 5.4 of [23]:

▶ **Corollary 32.** *Suppose $\mathcal{M}$ is C^2 and $p \in \mathcal{M}$, then for all $0 < r < \frac{\mathrm{rch}(\mathcal{M})}{\sqrt{2}}$, the projection $\pi_{T_p\mathcal{M}}$ onto the tangent space $T_p\mathcal{M}$, restricted to $\mathcal{M} \cap B(p, r)$ is a diffeomorphism onto its image.*

Proof of Corollary 32. Let $q \in \mathcal{M}$ such that $|p-q| \leq r$, then the differential of the projection map $\pi_{T_p\mathcal{M}}$ at q is non-degenerate, because, by Lemma 2, the angle $\angle(T_p\mathcal{M}, T_q\mathcal{M})$ is less than $\pi/2$. Because $\mathcal{M} \cap B(p, r)$ is a topological ball of the right dimension by Proposition 1 of [10], the result now follows. ◀

Similarly to Lemma 2 we have for $C^{1,1}$ manifolds that:

▶ **Lemma 33** (Theorem 3 of [10]). *Now suppose that $\mathcal{M}$ has positive reach, that is $\mathcal{M}$ is at least $C^{1,1}$, and let $|p - q| \leq \mathrm{rch}(\mathcal{M})/3$, then*

$$\sin \frac{\angle(T_p\mathcal{M}, T_q\mathcal{M})}{2} \leq \frac{1 - \sqrt{1 - \alpha^2}}{\sqrt{\frac{\alpha^2}{4} - (\frac{\alpha^2}{2} + 1 - \sqrt{1 - \alpha^2})^2}},$$

where $\alpha = |p - q|/\mathrm{rch}(\mathcal{M})$.

This lemma gives us a corollary, which is the equivalent of Corollary 32:

▶ **Corollary 34.** *Suppose $\mathcal{M}$ is $C^{1,1}$ and $p \in \mathcal{M}$, then for all $r < \frac{\mathrm{rch}(\mathcal{M})}{3}$, the projection $\pi_{T_p\mathcal{M}}$ onto the tangent space $T_p\mathcal{M}$, restricted to $\mathcal{M} \cap B(p, r)$ is a diffeomorphism onto its image.*

These are in fact all the fundamental results that are needed to be able to extend to the $C^{1,1}$ setting.

Assuming the manifold is $C^{1,1}$ would lead to minor changes in the calculations in the proof of Lemma 24 and would in theory influence the final conclusion in Section 7.2. However, because we have a significant margin in the difference between $\pi/2$ minus the angle between $\mathcal{N}_{\tilde{p}}$ and $N_p\mathcal{M}$ we would not need to change the constants in Section 7.2. The rest of proofs hold verbatim.

C Remark on the volume of $\mathcal{S}$

▶ **Remark 35.** *Because of symmetry the largest volume $\mathcal{S}$ can attain is when both delimiting hyperplanes are equidistant to the center of $B^d(r)$. The volume of $\mathcal{S}$ is given by the integral*

$$r^d \int_{-\rho_1}^{\rho_1} \mathrm{vol}\left(B_{d-1}\left(\sqrt{1-h^2}\right)\right) \mathrm{d}h = \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} r^d \int_{-\rho_1}^{\rho_1} \left(\sqrt{1-h^2}\right)^{d-1} \mathrm{d}h,$$

where $B_{d-1}(r)$ denotes the ball on dimension $d-1$ with radius r and Γ denotes the Euler gamma function. This integral can be expressed using special functions such as the hypergeometric function or beta functions. This gives an explicit value for $\bar{\rho}_1$.

D Notation

In the following table we give an overview of the notation used in this paper and compare it to Whitney's notation.

Notation	definition	Whitney's notation (if relevant)
$\mathcal{M}$	The manifold	M
$T\mathcal{M}$, $T_p\mathcal{M}$	The tangent bundle and the tangent space at p	P_p
$N\mathcal{M}$, $N_p\mathcal{M}$	The normal bundle and normal space at p	
$N_{\leq k}$	An upper bound on the total number of faces of dimension less or equal to k that contain a given vertex.	Whitney does not distinguish dimensions and uses N as an upper bound. (no value given)
$\mathrm{rch}(\mathcal{M})$	The reach to the manifold $\mathcal{M}$	
$\mathcal{S}$	Slab between two hyperplanes intersected with a ball	Q'
$\mathcal{T}$	The ambient Coxeter triangulation of type $\tilde{A}$	L is the ambient triangulation, but is not a Coxeter triangulation
$\tilde{\mathcal{T}}$	Perturbed ambient triangulation	L^*
v_i	Vertices of $\mathcal{T}$	p_i
v_i^*	Vertices of $\tilde{\mathcal{T}}$	p_i^*
$L(\cdot)$	Longest edge length	δ is the longest edge length of the ambient triangulation L
$U(X, r)$	A neighbourhood of radius r of a set X	$U_r(X)$
$\bar{\rho}_1$	Volume fraction of the part of a ball inside a slab	ρ_1
ρ_1	Lower bound on $\bar{\rho}_1$, see (4)	
δ	Protection	
μ	Separation as in an (ϵ, μ)-net (the shortest edge length in $\mathcal{T}$ for Coxeter triangulations)	
ϵ	The sampling density as in an (ϵ, μ)-net (the circumradius of the simplices in the Coxeter triangulation)	
μ_0	The normalized separation, that is $\mu = \mu_0\epsilon$	

A_i	Affine subspaces	P, P' and Q
$L(K)$	For any complex K, $L(K)$ denotes the longest edge length in K	
L	$L = L(\mathcal{T})$	
$\tilde{L}$	$\tilde{L} = L(\tilde{\mathcal{T}})$	
$\tilde{c}L$	Perturbation radius of the vertices of $\mathcal{T}$	ρ
$\tilde{c}$	Normalized perturbation radius	ρ^*
τ, σ	Simplices. We have tried to reserve τ for $\mathcal{T}$ or $\tilde{\mathcal{T}}$ and σ for K. However for arbitrary simplices (such as in Appendix A) we use arbitrary choices. Subscripts are used for indices and superscripts for the dimension.	Same
$t(\sigma)$	Thickness of σ	
aff	The affine hull	
n	Dimension of $\mathcal{M}$	n
d	Ambient dimension ($\mathbb{R}^d$)	m
$d(\cdot,\cdot)$	Euclidean distance between sets	
$d_{\mathcal{M}}(\cdot,\cdot)$	Distance on $\mathcal{M}$	
λ	barycentric coordinates	
K	Triangulation of $\mathcal{M}$	K
$B^d(c,r)$	A ball with in $\mathbb{R}^d$ of dimension with centre c and radius r, if we do not need to emphasize the centre or radius or they are to be determined, these are suppressed from the notation	$U_r(c)$
$B_{T_p\mathcal{M}}(c,r)$	A ball in $T_p\mathcal{M}$, using the same conventions as for $B^d(c,r)$	
$\mathring{C}(T_p\mathcal{M}, r_1, r_2)$	Open cylinder given by all points that project orthogonally onto an open ball of radius r_1 in $T_p\mathcal{M}$ and whose distance to this ball is at most r_2	
$\pi_{\mathcal{M}}$	Closest point projection on $\mathcal{M}$	π^*
$\pi_{T_p\mathcal{M}}$	Orthogonal projection on the tangent spaces $T_p\mathcal{M}$	
π_p^{-1}	See Definition 4	
$\pi_{\mathrm{aff}(\tau_k^{d-n})\to N_p\mathcal{M}}$ $= \pi_{\tau_k^{d-n}}$	The orthogonal projection map from the affine hull $\mathrm{aff}(\tau_i^{d-n})$ to $N_p\mathcal{M}$.	
$N_{v(\tau)}(e_i)$	See (20)	
$\mathcal{N}_{\bar{p}}$	The 'normal' space of K at $\bar{p}$, that is $\mathrm{span}(N_{\bar{p}}(e_i))$	

Overview most important bounds

α_1 and α_k have been defined by a recursion relation as follows

$$\alpha_1 = \frac{4}{3}\rho_1\tilde{c} \qquad \frac{2}{3}\alpha_{k-1}\tilde{c}\rho_1 = \alpha_k, \tag{5}$$

and thus $\alpha_k = \frac{2^{k+1}}{3^k}\rho_1^k\tilde{c}^k$. In particular we have the bound

$$\alpha_k \leq \frac{1}{18^k}. \tag{6}$$

L satisfies

$$\left(1-\sqrt{1-\left(\frac{6L(\mathcal{T})}{\mathrm{rch}(\mathcal{M})}\right)^2}\right)\mathrm{rch}(\mathcal{M})=\frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}L \tag{8}$$

or equivalently

$$\frac{L}{\mathrm{rch}(\mathcal{M})}=\frac{2\frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}}{\left(\frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}\right)^2+6^2}, \tag{9}$$

with

$$\zeta=\frac{8}{15\sqrt{d}\binom{d}{d-n}\cdot(1+2\tilde{c})}\left(1-\frac{8\tilde{c}}{t(\mathcal{T})^2}\right)t(\mathcal{T}). \tag{7}$$

We often use

$$\frac{L}{\mathrm{rch}(\mathcal{M})}<\frac{(\alpha_{d-n})^{4+2n}}{54(n+1)^2}\zeta^n<\frac{(\alpha_{d-n})^2}{54}, \qquad \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}<\frac{(\alpha_{d-n})^2}{3}\leq\frac{\alpha_{d-n}}{3}. \tag{10}$$

The perturbation radius $\tilde{c}$ satisfies

$$|v_i-\tilde{v}_i|\leq\tilde{c}L=\min\left\{\frac{t(\mathcal{T})\mu_0}{18d}\delta,\frac{1}{24}t(\mathcal{T})^2L\right\}, \tag{14}$$

from which it follows that

$$\tilde{c}\leq\frac{1}{24}. \tag{15}$$

E Proofs

Proof of Lemma 3. Apart from Lemma 2, we'll be using the following results from [10]: For a minimizing geodesic γ on $\mathcal{M}$ with length ℓ parametrized by arc length, with $\gamma(0)=p$ and $\gamma(\ell)=q$

$$\angle\dot{\gamma}(0)\dot{\gamma}(t)\leq\frac{t}{\mathrm{rch}(\mathcal{M})}. \tag{24}$$

If we also write $v_p=\dot{\gamma}(0)$ we have that

$$\begin{aligned}
\langle\gamma(\ell),v_p\rangle&=\int_0^\ell\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\gamma(t),v_p\rangle\right)\mathrm{d}t\\
&=\int_0^\ell\langle\dot{\gamma}(t),t_0\rangle\mathrm{d}t\\
&\geq\int_0^\ell\cos\left(\frac{t}{\mathrm{rch}(\mathcal{M})}\right)\mathrm{d}t && \text{(using (24))}\\
&=\mathrm{rch}(\mathcal{M})\sin\left(\frac{\ell}{\mathrm{rch}(\mathcal{M})}\right)\\
&\geq\mathrm{rch}(\mathcal{M})\sin(\angle(T_p\mathcal{M},T_q\mathcal{M})), && \text{(using Lemma 2)}
\end{aligned}$$

as long as $\ell<\frac{1}{2}\mathrm{rch}(\mathcal{M})\pi$. Because $v_p\in T_p\mathcal{M}$ and $\gamma(\ell)=q$ we have that

$$|p-\pi_{T_p\mathcal{M}}(q)|\geq\mathrm{rch}(\mathcal{M})\sin(\angle(T_p\mathcal{M},T_q\mathcal{M})). \tag{25}$$

This means in particular that for all q such that $|p-\pi_{T_p\mathcal{M}}(q)| < \mathrm{rch}(\mathcal{M})$ and $|q-\pi_{T_p\mathcal{M}}(q)| \leq \mathrm{rch}(\mathcal{M})$ the angle between $T_p\mathcal{M}$ and $T_q\mathcal{M}$ is less than 90 degrees. Note that the condition on ℓ mentioned above is satisfied by a combination of Theorem 1 and Lemma 11 of [10]. ◀

Proof of Claim 10. Choudhary et al. [13, Appendix B] provide explicit values of all the quantities mentioned in Corollary 9 for a Coxeter triangulation of type $\tilde{A}$, with the exception of μ, which can be easily derived from a more general result. If we fix the scale (which in [13] we did by a convenient choice of coordinates for the vertices), they are

$$L(\sigma) = \begin{cases} \frac{\sqrt{d+1}}{2} & \text{if } d \text{ is odd,} \\ \frac{1}{2}\sqrt{\frac{d(d+2)}{(d+1)}} & \text{if } d \text{ is even} \end{cases} \qquad t(\sigma) = \begin{cases} \sqrt{\frac{2}{d}} & \text{if } d \text{ is odd,} \\ \sqrt{\frac{2(d+1)}{d(d+2)}} & \text{if } d \text{ is even} \end{cases}$$

$$\epsilon = \sqrt{\frac{d(d+2)}{12(d+1)}} \qquad \delta(\sigma) = \frac{\sqrt{d^2+2d+24} - \sqrt{d^2+2d}}{\sqrt{12(d+1)}}. \tag{26}$$

The value of μ easily follows from the general expression of edge lengths (see [13, Appendix B, $\tilde{A}_d$, item 5]) and is equal to $\mu = \sqrt{\frac{d}{d+1}}$. From (26), we get that $\mu_0 = \frac{\mu}{\epsilon} = \sqrt{\frac{12}{d+2}}$. The bound in (2) is therefore

$$\tilde{c} \leq \frac{t(\sigma)\mu_0}{18d}\frac{\delta}{L} = \begin{cases} \sqrt{2}\frac{\sqrt{d^2+2d+24}-\sqrt{d^2+2d}}{9d^{3/2}\sqrt{d+2}(d+1)} & \text{if } d \text{ is odd,} \\ \sqrt{2(d+1)}\frac{\sqrt{d^2+2d+24}-\sqrt{d^2+2d}}{9d^2(d+2)^{3/2}} & \text{if } d \text{ is even.} \end{cases}$$

$$\leq \sqrt{2}\frac{\sqrt{d^2+2d+24}-\sqrt{d^2+2d}}{9d^{3/2}\sqrt{d+2}(d+1)} \sim \frac{\sqrt{32}}{3d^4},$$

where we used that $\sqrt{1+x} \simeq 1 + \frac{1}{2}x$ if x is close to zero. ◀

Proof of lemma 12. We first notice that the barycentre c_b of a simplex σ^j is at least a distance $\min \mathrm{alt}(\sigma^j)/(j+1)$ removed from the faces of the simplex. This means that the ball in $\mathrm{aff}(\sigma^j)$ centred on c with radius $\min \mathrm{alt}(\sigma^j)/(j+1)$, denoted by $B_{\mathrm{aff}(\sigma^j)}(c, \min \mathrm{alt}(\sigma^j)/(j+1))$, is contained in σ^j. We now consider any line segment ℓ connecting a pair of antipodal points of $\partial B_{\mathrm{aff}(\sigma^j)}(c, \min \mathrm{alt}(\sigma^j)/(j+1))$. This line segment is contained in a $d_{\max}$ neighbourhood of A_0 and thus

$$\sin \angle(\ell, A_0) \leq \frac{(j+1)\, d_{\max}}{\min \mathrm{alt}(\sigma)}.$$

The result now follows, because ℓ is arbitrarily chosen. ◀

Proof of Claim 15. Theorem 3 [24] gives us that for $d \geq 2$ and $1 \leq j \leq d-1$

$$\frac{1}{2}(j^2+j+2)j^{d-j-1} - 1 \leq S(d,j) \leq \frac{1}{2}\binom{d}{j}j^{d-j}.$$

Furthermore, Stirlings theorem and the binomial theorem give that $j! \sim j^j$ and $\sum_{j=0}^{k}\binom{d}{j} \lesssim d^k$, respectively. We now see that

$$N_{\leq k} = 2 + \sum_{j=1}^{k} j! S(d+1,j) \lesssim \sum_{j=1}^{k} j! \binom{d+1}{j} j^{d+1-j} \lesssim k^d \sum_{j=1}^{k}\binom{d}{j} \lesssim k^d d^k.$$

It is clear that if k is much smaller than d that then k^d dominates. ◀

Proof of Lemma 16. We can bound the volume of the slab $\mathcal{S}$ by the cylinder with base $B_{d-1}(r)$ and height $2\rho_1 r$, that is

$$2\rho_1 r^d \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})}.$$

This means that

$$\begin{aligned}\frac{\mathrm{vol}(\mathcal{S})}{\mathrm{vol}(B^d(r))} &< \frac{2\rho_1 r \mathrm{vol}(B_{d-1}(r))}{\mathrm{vol}(B^d(r))}\\ &= \frac{2\rho_1 \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2}+1)}}{\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}}\\ &= \frac{2\rho_1\Gamma\left(\frac{d}{2}+1\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}+1\right)}\\ &= \begin{cases}\frac{\pi(2k)!}{2^{2k-1}(k!)^2}\rho_1 & \text{if } d=2k\\ \frac{2^{2k+1}k!(k-1)!}{(2k)!}\rho_1 & \text{if } d=2k-1.\end{cases}\end{aligned}$$

using the standard formulae for the volume of the ball, see for example [16, page 622]. Note that the inequality is strict because $\rho_1 > 0$. We see that therefore ρ_1 may be chosen to be

$$\rho_1 = \begin{cases}\frac{2^{2k-2}(k!)^2}{\pi(2k)!N_{\leq d-n-1}} & \text{if } d=2k\\ \frac{(2k)!}{2^{2k+2}k!(k-1)!N_{\leq d-n-1}} & \text{if } d=2k-1.\end{cases} \tag{4}$$

From Wendel's bound on the ratio of Gamma functions [25], we immediately see that for a fixed constant a,

$$\frac{\Gamma(x+a)}{\Gamma(x)} \sim x^a.$$

This means that

$$\frac{2\rho_1\Gamma\left(\frac{d}{2}+1\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}+1\right)} \sim \frac{2\rho\left(\frac{d}{2}+\frac{1}{2}\right)^{1/2}}{\sqrt{\pi}} \sim \sqrt{d}.$$

We now see that

$$\rho_1 \sim \frac{1}{\sqrt{d}N_{\leq d-n-1}}.$$

◀

Proof of Claim 18. Because we assume that the perturbation is sufficiently small compared to the protection, as given in the first condition of (14), (2) is satisfied and $\tilde{\mathcal{T}}$ will have exactly the same combinatorial structure as $\mathcal{T}$.

By the third condition of (14) we have a lower bound on the quality of the simplices. To be precise, we have that for any simplex τ in $\tilde{\mathcal{T}}$

$$t(\tau) \geq \frac{4}{5\sqrt{d}}\left(1 - \frac{8\tilde{c}}{t(\mathcal{T})^2}\right)t(\mathcal{T}), \tag{13}$$

as a consequence of Lemma 11, the fact that if you perturb the vertices by $\tilde{c}L$ the edge lengths are perturbed $2\tilde{c}$ (that is $2\tilde{c} = c_0$) and the fact that if $\sigma \subset \tau$, then $t(\sigma) \geq t(\tau)$. So we have established (13). ◀

Proof of Claim 19. The argument is volumetric. Let us first introduce the notation $U(X, r)$ for the set of all points $x \in \mathbb{R}^d$ such that $d(x, X) \leq r$, where X is any subset of $\mathbb{R}^d$. By definition of ρ_1, see 'Balls and exclusion volumes' in Section 5.1, and because the dimension of $\operatorname{span}(\tau'_j, T_p\mathcal{M})$ is at most $d-1$, we have that

$$\operatorname{vol}(B(v_i, \tilde{c}L) \cap U(\operatorname{span}(\tau'_j, T_p\mathcal{M}), \rho_1\tilde{c}L)) \leq \frac{\operatorname{vol}(B^d(r))}{2N_{\leq d-n-1}}.$$

It now follows that

$$\begin{aligned}
&\operatorname{vol}\left(B(v_i, \tilde{c}L) \Big\backslash \left(\bigcup_{1\leq j\leq \nu} U(\operatorname{span}(\tau'_j, T_p\mathcal{M}), \rho_1\tilde{c}L)\right)\right) \\
&\quad \geq \operatorname{vol}(B(v_i, \tilde{c}L)) - \sum_{0\leq j\leq \nu} \operatorname{vol}(B(v_i, \tilde{c}L) \cap U(\operatorname{span}(\tau'_j, T_p\mathcal{M}), \rho_1\tilde{c}L)) \\
&\quad > \operatorname{vol}(B(v_i, \tilde{c}L)) - \sum_{0\leq j\leq \nu} \frac{\operatorname{vol}(B(v_i, \tilde{c}L))}{2N_{\leq d-n-1}} \\
&\quad = B(v_i, \tilde{c}L)\left(1 - \frac{\nu+1}{2N_{\leq d-n-1}}\right) \\
&\quad \geq \frac{1}{2}B(v_i, \tilde{c}L),
\end{aligned}$$

where we used that $\nu \leq N_{\leq d-n-1}$ in the last line, by definition, as mentioned in the description of Case 2. Because the volume is positive we know there exists a point $\tilde{v}_i$ that satisfies

$$d(\tilde{v}_i, \operatorname{span}(\tau'_j, T_p\mathcal{M})) > \rho_1\tilde{c}L, \tag{27}$$

for all $1 \leq j \leq \nu$. ◀

Proof of Claim 20. We first make use of the induction[2] hypothesis $d(\tau'_j, \mathcal{M}) > \alpha_{k-1}L$ to find a bound on the distance from τ'_j to the tangent space $T_p\mathcal{M}$, then bound the distance from $\tilde{v}_i * \tau'_j = \tau_j$ to $T_p\mathcal{M}$ based on this. For this argument to work, we have to assume that τ'_j is not the empty set, that is $j \neq 0$. This case is handled separately at the end.

Because $d(\tau'_j, \mathcal{M}) > \alpha_{k-1}L$ and the ball in the tangent space $B_{T_p\mathcal{M}}(p, r)$ centred on p of radius $6L = r$, satisfies

$$B_{T_p\mathcal{M}}(p, r) \subset U\left(\mathcal{M}, \left(1 - \sqrt{1 - \left(\frac{r}{\operatorname{rch}(\mathcal{M})}\right)^2}\right)\operatorname{rch}(\mathcal{M})\right).$$

Thanks to Lemma 6, we have that

$$d(\tau'_j, B_{T_p\mathcal{M}}(p, r)) > \alpha_{k-1}L - \left(1 - \sqrt{1 - \left(\frac{r}{\operatorname{rch}(\mathcal{M})}\right)^2}\right)\operatorname{rch}(\mathcal{M}).$$

[2] In particular $\tau'_j \subset \tilde{\mathcal{T}}_i$.

This can be simplified:

$$\begin{aligned}
&d(\tau_j', B_{T_p\mathcal{M}}(p,r)) \\
&\quad > \alpha_{k-1}L - \left(1 - \sqrt{1 - \left(\frac{r}{\mathrm{rch}(\mathcal{M})}\right)^2}\right)\mathrm{rch}(\mathcal{M}) \\
&\quad > \alpha_{k-1}L - \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}L && \text{(using (8))} \\
&\quad > \alpha_{k-1}L - \frac{1}{3}\alpha_{d-n}L && \text{(using (10))} \\
&\quad \geq \frac{2}{3}\alpha_{k-1}L. && \text{(because } \alpha_{k-1} > \alpha_k\text{)}
\end{aligned} \tag{28}$$

Because $d(v_i, p) < \frac{3}{2}L$ and $\tilde{L} < L + 2\tilde{c}L$, and $\tilde{c} < \frac{1}{24}$, see (15), we have the very course bound that

$$d(\tau_j', p) \leq 4L, \tag{29}$$

by the triangle inequality. We thus find that

$$d(\tau_j', T_p\mathcal{M} \setminus B_{T_p\mathcal{M}}(p,r)) > 2L.$$

This means that (28) holds for the entire tangent space, that is,

$$d(\tau_j', T_p\mathcal{M}) > \frac{2}{3}\alpha_{k-1}L. \tag{30}$$

Lemma 31, with $A_1 = T_p\mathcal{M}$ and $A_2 = \mathrm{span}(\tau_j', T_p\mathcal{M})$, now gives

$$d(\tau_j, T_p\mathcal{M}) \geq \frac{d(\tau_j', T_p\mathcal{M})d(v_i, \mathrm{span}(\tau_j', T_p\mathcal{M}))}{L + 2\tilde{c}L}.$$

This can again be simplified

$$\begin{aligned}
d(\tau_j, T_p\mathcal{M}) &\geq \frac{d(\tau_j', T_p\mathcal{M})d(v_i, \mathrm{span}(\tau_j', T_p\mathcal{M}))}{L + 2\tilde{c}L} \\
&> \frac{\left(\frac{2}{3}\alpha_{k-1}L\right)\rho_1\tilde{c}L}{L + 2\tilde{c}L} && \text{(thanks to (30) and (17))} \\
&> \frac{\frac{2}{3}L}{\frac{4}{3}L}\alpha_{k-1}\rho_1\tilde{c}L && \text{(because } \tilde{c} \leq \tfrac{1}{24}\text{)} \\
&= \frac{4}{3}\alpha_k L. && \text{(using the relation (5) for } \alpha_k\text{)}
\end{aligned} \tag{31}$$

Similarly to (29), we have that

$$d(\tau_j, p) \leq 4L < 6L.$$

We can go from the distance from τ_j to the tangent space, as given in (31), to the distance to the manifold as follows. Because of Lemma 5 we can localize the results and Lemma 6 allows us to estimate the difference in distance to the manifold and the tangent space. This gives

$$d(\tau_j, \mathcal{M}) > \frac{4}{3}\alpha_k L - \left(1 - \sqrt{1 - \left(\frac{6L}{\mathrm{rch}(\mathcal{M})}\right)^2}\right) \mathrm{rch}(\mathcal{M}).$$

This can be again simplified

$$\begin{aligned} d(\tau_j, \mathcal{M}) &> \frac{4}{3}\alpha_k L - \left(1 - \sqrt{1 - \left(\frac{6L}{\mathrm{rch}(\mathcal{M})}\right)^2}\right) \mathrm{rch}(\mathcal{M}) \\ &> \frac{4}{3}\alpha_k L - \frac{1}{3}\alpha_{d-n} L && \text{(using (8) and (10))} \\ &\geq \alpha_k L. && \text{(because } \alpha_k \geq \alpha_{d-n} \text{ if } k \leq d-n-1 \text{ by (5))} \end{aligned}$$

This completes the proof for the case where $j \neq 0$ or τ_j is non-empty.

For $j = 0$, (27) and Lemma 6 yield

$$d(\tau_j, \mathcal{M}) > \rho_1 \tilde{c} L - \left(1 - \sqrt{1 - \left(\frac{6L}{\mathrm{rch}(\mathcal{M})}\right)^2}\right) \mathrm{rch}(\mathcal{M}).$$

We simplify

$$\begin{aligned} d(\tau_j, \mathcal{M}) &> \rho_1 \tilde{c} L - \left(1 - \sqrt{1 - \left(\frac{6L}{\mathrm{rch}(\mathcal{M})}\right)^2}\right) \mathrm{rch}(\mathcal{M}). \\ &> \rho_1 \tilde{c} L - \frac{1}{3}\alpha_{d-n} L && \text{(using (8) and (10))} \\ &> \alpha_1 L. && \text{(by definition of (5))} \end{aligned}$$

◀

The following proof differs from Whitney's proof.

Proof of Lemma 21. Because $\tau^k \subset B(p, 6L)$, the point in $T_p\mathcal{M}$ that is closest to τ lies in $T_p\mathcal{M} \cap B(p, 6L) = B_{T_p\mathcal{M}}(p, 6L)$. We now see that

$$\begin{aligned} d(\tau^k, T_p\mathcal{M}) &\geq d(\tau^k, B_{T_p\mathcal{M}}(p, 6L)) && \text{(first sentence of the proof)} \\ &> d(\tau^k, \mathcal{M}) - \left(1 - \sqrt{1 - \left(\frac{6L}{\mathrm{rch}(\mathcal{M})}\right)^2}\right) \mathrm{rch}(\mathcal{M}) && \text{(Lemma 6)} \\ &> \alpha_k L - \frac{1}{3}\alpha_{d-n} L, && (d(\tau^k, \mathcal{M}) > \alpha_k L \text{ and (10))} \\ &> \frac{2}{3}\alpha_k L, && (\alpha_k \geq \alpha_{d-n} \text{ for } k \leq d-n) \end{aligned}$$

which completes the proof. ◀

Proof of Lemma 22. Let $\bar{p} \in \mathcal{M} \cap \tau^k$. Lemma 3 (and (8), (10)) gives us that $\bar{p} \in \pi_p^{-1}$ $(B_{T_p\mathcal{M}}(p, 6L))$, where we use the notation of Definition 4. Lemma 6 gives that

$$d(\bar{p}, T_p\mathcal{M}) \leq \left(1 - \sqrt{1 - \left(\frac{6L}{\mathrm{rch}(\mathcal{M})}\right)^2}\right) \mathrm{rch}(\mathcal{M}) < \frac{1}{3}\alpha_{d-n} L.$$

Let $\check{\tau} \subset \tau^k$ be the face of smallest dimension such that $d(\check{\tau}, T_p\mathcal{M}) \leq \frac{2}{3}\alpha_{d-n}L$. This face exists thanks to the triangle inequality. By Lemma 21 we have the $\dim(\check{\tau}) \geq d-n$. Lemma 30 implies that $\check{\tau}$ intersects $T_p\mathcal{M}$. The reason for this is the following; $\check{\tau}$ is the simplex of the smallest dimension such that $d(\check{\tau}, T_p\mathcal{M}) \leq \frac{2}{3}\alpha_k L$ meaning in particular that $d(\check{\tau}, T_p\mathcal{M}) < d(\partial\check{\tau}, T_p\mathcal{M})$. Because $\check{\tau}$ is a face of τ^k, clearly $T_p\mathcal{M}$ intersects τ^k. ◀

Proof of Lemma 23. This is an immediate consequence of Lemma 30, (14), and the previous lemmas. ◀

Proof of Lemma 24. Let $\check{\tau}$ be a face of smallest dimension of τ^k such that $d(\check{\tau}, T_p\mathcal{M}) \leq \frac{2}{3}\alpha_n L$. Now Lemma 30 and Lemma 22 give that $\check{\tau}$ and $T_p\mathcal{M}$ have a unique point $\bar{p}$ in common and the dimension of $\check{\tau}$ is $d-n$.

Thanks to Lemma 3, $\mathcal{M}$ can be written as the graph of a function f, in a neighbourhood of at most size $\mathrm{rch}(\mathcal{M})$. We note that $f : T_p\mathcal{M} \simeq \mathbb{R}^n \to N_p\mathcal{M} \simeq \mathbb{R}^{d-n}$, where here we really think of the tangent and normal spaces as embedded in $\mathbb{R}^d$. Using the identification of $T_p\mathcal{M}$ with $\mathbb{R}^n$, we now define

$$F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^d : (\lambda, x) \mapsto (x, \lambda f(x)).$$

Note that $F(0, \cdot)$ gives a parametrization of $T_p\mathcal{M}$. Similarly, we can define $G : \mathbb{R}^{d-n} \to \mathbb{R}^d$ to be a linear (orthonormal) parametrization of $\mathrm{aff}(\check{\tau})$. We now consider the difference of the two functions $F - G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{d-n} = \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Thanks to Lemma 23 we have that

$$\sin \angle(\mathrm{aff}(\check{\tau}), T_p\mathcal{M}) \geq \frac{16}{10}\alpha_{d-n}.$$

Lemma 2, and (10) give that for any $q \in B(p, 6L)$

$$\sin\left(\frac{\angle(T_p\mathcal{M}, T_q\mathcal{M})}{2}\right) \leq \frac{6L}{2\mathrm{rch}(\mathcal{M})} \leq \frac{6}{2} \cdot \frac{(\alpha_{d-n})^2}{54} = \frac{1}{18}(\alpha_{d-n})^2.$$

It is clear that this also gives an upper bound on the angle between $T_p\mathcal{M}$ and the graph of $F(\lambda, \cdot)$ (denoted by $\mathrm{graph}\, F(\lambda, \cdot)$)for all $\lambda \in [0,1]$, due to linearity of the inner product. Because the upper bound on the angle between the tangent spaces is much smaller than the lower bound on $\angle(\mathrm{aff}(\tau^k), T_p\mathcal{M})$, $\mathrm{aff}(\check{\tau})$ and $T_q\, \mathrm{graph}\, F(\lambda, \cdot)$ span $\mathbb{R}^d$, for any $\lambda \in [0,1]$ and $q \in B(p, 6L)$. The implicit function theorem and the fact that $\check{\tau}$ and $T_p\mathcal{M}$ have a unique point $\bar{p}$ in common now give that the intersection $\bar{p}_\lambda$ between $\mathrm{graph}\, F(\lambda, \cdot) \cap B(p, 6L)$ and $\mathrm{aff}(\check{\tau})$ exists and is unique, for all $\lambda \in [0,1]$.

We can now use Lemmas 3, 6, and 23, to bound $|\bar{p} - \bar{p}_\lambda|$. The distance from the manifold to the tangent space (and thus the same holds for $\mathrm{graph}\, F(\lambda, \cdot)$) is bounded from above by

$$\frac{(\alpha_{d-n})^{3+2n}}{3(n+1)}\zeta^{2n}L < \frac{1}{3}(\alpha_{d-n})^2 L,$$

due to (8), and (10), while $\sin \angle(\mathrm{aff}(\check{\tau}), T_p\mathcal{M}) \geq \frac{16}{10}\alpha_{d-n}$, so

$$|\bar{p} - \bar{p}_\lambda| \leq \frac{\frac{1}{3}(\alpha_{d-n})^2 L}{\frac{16}{10}\alpha_{d-n}} \leq \frac{1}{3}\alpha_{d-n}L.$$

This distance bound is smaller than the distance bound of $\bar{p}$ to the boundary of $\check{\tau}$, due to Lemma 21. This means that $\bar{p}_\lambda \in \check{\tau}$, and in particular that $\mathcal{M}$ intersects τ^k. ◀

Proof of Lemma 25. Take $p \in \mathcal{M} \cap \tau$. Let $\check{\tau}^k$ be a face of the smallest dimension of τ, with $v_i \in \check{\tau}^k$, that intersects $T_p\mathcal{M}$. Now assume that $k > d-n$. Let us write $\check{\tau}^{k-1}$ for the face of $\check{\tau}^k$ opposite v_i. Because the dimension of $\check{\tau}^k \cap T_p\mathcal{M}$ is at least 1, the intersection of $T_p\mathcal{M}$ and $\check{\tau}^{k-1}$ is non-empty.

Similarly to the first argument in the proof of Lemma 24, we see that $T_p\mathcal{M}$ intersects some $(d-n)$-face of $\check{\tau}^{k-1}$. Thanks to Lemma 23 the angle between this $(d-n)$-face and $T_p\mathcal{M}$ is bounded from below. Due to Lemma 21 the intersection lies in the interior of the $(d-n)$-face. The angle bound and the fact that the intersection lies in the interior gives that any simplex in $\mathcal{T}$ that contains this $(d-n)$-face has points in the interior that lie in $T_p\mathcal{M}$. In particular the interior of $\check{\tau}^k$ contains part of $T_p\mathcal{M}$. Because both the interior of $\check{\tau}^k$ and $\check{\tau}^{k-1}$ contain points of $T_p\mathcal{M}$, linearity gives that $T_p\mathcal{M}$ must intersect $\partial\check{\tau}^k \setminus \check{\tau}^{k-1}$. From this contradiction of the assumption, we conclude that $k = d-n$.

Lemma 24 finally says that $\mathcal{M}$ intersects $\check{\tau}^k$, because $T_p\mathcal{M}$ does. ◀

Proof of Lemma 26. This inequality relies on estimates on the barycentric coordinates, and Lemma 11. We first establish a bound on the barycentric coordinates of $v(\tau_i^{d-n})$ for some $(d-n)$-dimensional simplex $\tau_i^{d-n} \in \tilde{\mathcal{T}}$ that intersects $\mathcal{M}$. By Lemma 21, $v(\tau_i^{d-n})$ lies at least a distance $\frac{2}{3}\alpha_{d-n-1}L)$ from the boundary $\partial\tau_i^{d-n}$ and the longest edge is at most $L + 2\tilde{c}L$. This means that all the barycentric coordinates λ_l with respect to (the vertices of) τ_i^{d-n} are at least

$$\lambda_l(\tau_i^{d-n}) > \frac{2}{3}\alpha_{d-n-1}\frac{L}{L+2\,\tilde{c}\,L} = \frac{2}{3}\alpha_{d-n-1}\frac{1}{1+2\,\tilde{c}}. \tag{32}$$

Let τ^d now be a top dimensional simplex in $\tilde{\mathcal{T}}$ that intersects $\mathcal{M}$. Let $\tau_1^{d-n}, \dots, \tau_j^{d-n}$ be the faces of τ^d that intersect $\mathcal{M}$. This means that $d-n+1$ barycentric coordinates with respect to τ^d of any $v(\tau_i^{d-n})$ satisfy the bound (32), while the other n coordinates are zero. This also means that for the barycentric coordinates with respect to τ^d of

$$v(\tau^k) = \frac{1}{j}\left(v(\tau_1^{d-n}) + \dots + v(\tau_j^{d-n})\right),$$

for $k > d-n$, we have that

- $k+1$ of the coordinates λ_l satisfy

$$\lambda_l > \frac{2}{3}\frac{1}{j}\frac{1}{1+2\,\tilde{c}}\alpha_{d-n-1}.$$

- The other $d-k$ coordinates are zero.

Note that $j \leq \binom{d}{d-n}$. This means that

$$d(v(\tau^k), \partial\tau^k) \geq \frac{2}{3\binom{d}{d-n}\cdot(1+2\,\tilde{c})}\alpha_{d-n-1}\min\operatorname{alt}(\tau^d).$$

We now have

$$\begin{aligned}
d(v(\tau^k), \partial\tau^k) &\geq \frac{2}{3\binom{d}{d-n}\cdot(1+2\,\tilde{c})}\alpha_{d-n-1}t(\tilde{\mathcal{T}})\tilde{L} && \text{(by definition of the thickness)}\\
&\geq \frac{2}{3\binom{d}{d-n}\cdot(1+2\,\tilde{c})}\alpha_{d-n-1}\frac{4}{5\sqrt{d}}\left(1-\frac{8\tilde{c}}{t(\mathcal{T})^2}\right)t(\mathcal{T})\tilde{L} \\
& && \text{(by the estimate (13))}\\
&\geq \frac{8}{15\sqrt{d}\binom{d}{d-n}\cdot(1+2\,\tilde{c})}\left(1-\frac{8\tilde{c}}{t(\mathcal{T})^2}\right)\alpha_{d-n-1}t(\mathcal{T})\tilde{L}
\end{aligned}$$

Using this estimate and the fact that σ^n defined through a sequence $\tau_0^{d-n} \subset \tau_1^{d-n+1} \subset \cdots \subset \tau_n^d$ we can give a lower bound on the minimal altitude of the simplex.

We are going to use the following easy observation on the minimal altitude simplices. Suppose that:

- The simplex σ is the join of a point p and the simplex σ'.
- $d(p, \operatorname{aff}(\sigma')) \geq d_{\min}$.
- $\min \operatorname{alt}(\sigma') \geq h'$.
- The maximum edge length of σ is $L(\sigma)$.

Then the $\min \operatorname{alt}(\sigma) \geq \frac{h' d_{\min}}{L(\sigma)}$, as can be established by simple trigonometric arguments, as illustrated in Figure 3

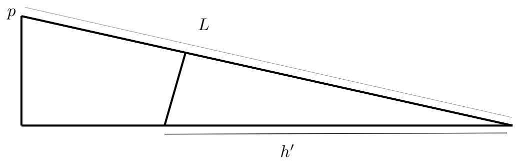

Figure 3 Both triangles are right angled.

This gives that $\min \operatorname{alt}(\sigma^n)$ with σ^n as in (18) is lower bounded as follows:

$$\begin{aligned} \min \operatorname{alt}(\sigma^n) &> \left(\frac{8}{15\sqrt{d}\binom{d}{d-n} \cdot (1 + 2\,\tilde{c})} \left(1 - \frac{8\tilde{c}}{t(\mathcal{T})^2}\right) \alpha_{d-n-1} t(\mathcal{T}) \right)^n \tilde{L} \\ &= \zeta^n (\alpha_{d-n-1})^n \tilde{L}, \end{aligned}$$

which completes the proof. ◀

Proof of Lemma 27. By Lemma 12 we have that

$$\sin \angle(\operatorname{aff}(\sigma^n), T_p\mathcal{M}) \leq \frac{(n+1)\, d_{\max}}{\min \operatorname{alt}(\sigma^n)},$$

where $d_{\max}$ denotes the maximum distance of the vertices of σ^n to $T_p\mathcal{M}$. Lemma 26 gives us the following bound

$$\min \operatorname{alt}(\sigma^n) > (\alpha_{d-n-1})^n \zeta^n \tilde{L}.$$

Finally, $d_{\max}$ is bounded thanks to (8). Combining these results yields

$$\begin{aligned} \sin \angle(\operatorname{aff}(\sigma^n), T_p\mathcal{M}) &\leq \frac{(n+1)\, \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2} \zeta^{2n} L}{(\alpha_{d-n-1})^n \zeta^n \tilde{L}} \\ &\leq \frac{(\alpha_{d-n})^{4+n}}{6(n+1)} \zeta^n. \end{aligned}$$

(because $\alpha_{d-n-1} < \alpha_{d-n}$, and $\tilde{L} \geq L$ because there are unperturbed simplices in $\tilde{\mathcal{T}}$)

◀

Proof of Lemma 28. We first establish a bound on the angles between $\mathcal{N}_{\bar{p}}$ and $\mathcal{N}_{\bar{q}}$. Lemma 23 gives that for each τ^{d-n}

$$\sin \angle(\operatorname{aff}(\tau^{d-n}), T_p\mathcal{M}) \geq \frac{16}{10} \alpha_{d-n},$$

so that

$$\cos\angle(\pi^{-1}_{\mathrm{aff}(\tau^{d-n})\to N_p\mathcal{M}}(e_j), e_j) \geq \frac{16}{10}\alpha_{d-n},$$

with e_i a basis vector of $N_p\mathcal{M}$ as in Section 7.1. This means that $|\pi^{-1}_{\mathrm{aff}(\tau^{d-n})\to N_p\mathcal{M}}(e_j)| \leq \frac{10}{16\alpha_{d-n}}$ and as a consequence of this and the triangle inequality we find that $|N_{v(\tau)}(e_j)|, |N_{\bar{p}'}(e_j)| \leq \frac{10}{16\,\alpha_{d-n}}$, for any $\bar{p}'$. This in turn gives us that

$$\angle(N_{\bar{p}'}(e_j), e_j) \leq \arccos\left(\frac{16}{10}\alpha_{d-n}\right). \tag{33}$$

The triangle inequality for angles (or points on the sphere) now implies that

$$\angle(N_{\bar{p}}(e_j), N_{\bar{q}}(e_j)), \angle(N_{v(\tau_i^{d-n})}(e_j), N_{v(\tau_l^{d-n})}(e_j)) \leq 2\arccos\left(\frac{16}{10}\alpha_{d-n}\right).$$

We can tighten this result for $\angle(N_{\bar{p}}(e_j), N_{\bar{q}}(e_j))$ by the use of the barycentric coordinates. We now consider the e_i component of both $\lambda_0 N_{v(\tau_0^{d-n})}(e_i) + \cdots + \lambda_n N_{v(\tau_n^{d-n})}(e_i)$ and $\lambda'_0 N_{v(\tau_0^{d-n})}(e_i) + \cdots + \lambda'_n N_{v(\tau_n^{d-n})}(e_i))$ is e_i, by construction. We are going to compare this with the length of $\lambda_0 N_{v(\tau_0^{d-n})}(e_i) + \cdots + \lambda_n N_{v(\tau_n^{d-n})}(e_i)$ and $\lambda'_0 N_{v(\tau_0^{d-n})}(e_i) + \cdots + \lambda'_n N_{v(\tau_n^{d-n})}(e_i)$. For estimates on these lengths we need to introduce the following notation: $(N_{v(\tau_0^{d-n})}(e_i) \dots N_{v(\tau_n^{d-n})}(e_i))$ denotes the matrix whose columns are the vectors $N_{v(\tau_0^{d-n})}(e_i), \dots, N_{v(\tau_n^{d-n})}(e_i)$, $\|\cdot\|_2$ denotes the operator 2-norm, and $\|\cdot\|_F$ the Frobenius norm. With this notation we can now derive the following bound:

$$\begin{aligned}
&|\lambda_0 N_{v(\tau_0^{d-n})}(e_i) + \cdots + \lambda_n N_{v(\tau_n^{d-n})}(e_i) - (\lambda'_0 N_{v(\tau_0^{d-n})}(e_i) + \cdots + \lambda'_n N_{v(\tau_n^{d-n})}(e_i))| \\
&\quad = |(\lambda_0 - \lambda'_0) N_{v(\tau_0^{d-n})}(e_i) + \cdots + (\lambda_n - \lambda'_n) N_{v(\tau_n^{d-n})}(e_i)| \\
&\quad = \left| (N_{v(\tau_0^{d-n})}(e_i) \dots N_{v(\tau_n^{d-n})}(e_i)) \begin{pmatrix} \lambda_0 - \lambda'_0 \\ \vdots \\ \lambda_n - \lambda'_n \end{pmatrix} \right| \\
&\quad \leq \|(N_{v(\tau_0^{d-n})}(e_i) \dots N_{v(\tau_n^{d-n})}(e_i))\|_2 |\lambda - \lambda'| \\
&\quad \leq \|(N_{v(\tau_0^{d-n})}(e_i) \dots N_{v(\tau_n^{d-n})}(e_i))\|_F |\lambda - \lambda'| \qquad \text{(because } \|\cdot\|_2 \leq \|\cdot\|_F\text{)} \\
&\quad = \sqrt{|N_{v(\tau_0^{d-n})}(e_i)|^2 + \cdots + |N_{v(\tau_n^{d-n})}(e_i)|^2}\, |\lambda - \lambda'| \\
&\quad \leq \frac{10\sqrt{n+1}}{16\,\alpha_{d-n}} |\lambda - \lambda'|.
\end{aligned}$$

We now turn our attention to the triangle, with edges $N_{\bar{p}}(e_j)$, $N_{\bar{q}}(e_j)$, and $N_{\bar{p}}(e_j) - N_{\bar{q}}(e_j)$ as depicted in Figure 4. We apply the sine rule to this triangle and find

$$\sin\angle(N_{\bar{p}}(e_j), N_{\bar{q}}(e_j)) \leq \sin\phi_0 = \frac{\frac{10\sqrt{n+1}}{16\,\alpha_{d-n}}|\lambda - \lambda'|\frac{10}{16\,\alpha_{d-n}}}{1} = \left(\frac{10}{16\,\alpha_{d-n}}\right)^2 \sqrt{n+1}|\lambda - \lambda'|. \tag{34}$$

Note that this can be tightened at the cost of introducing extra square roots.

We now consider the triangle $\bar{p}\bar{q}x$ and we want to estimate $|\bar{p}x|$, and $|\bar{q}x|$. This following estimate will use:

- the sine rule

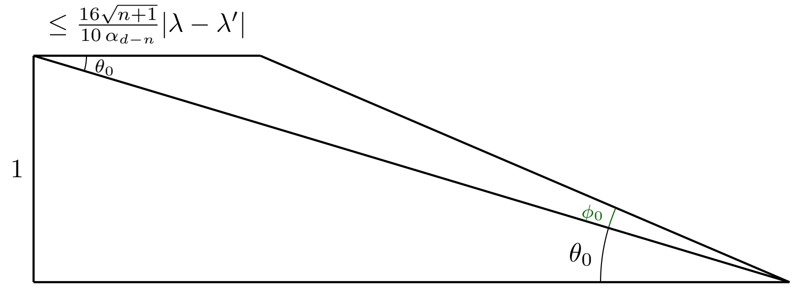

Figure 4 The worst case for the angle between the vectors $N_{\bar{p}}(e_j)$, and $N_{\bar{q}}(e_j)$. We write ϕ_0 for an upper bound on $\angle(N_{\bar{p}}(e_j), N_{\bar{q}}(e_j))$. Moreover $\theta_0 \geq \arccos(\frac{16}{10}\alpha_{d-n})$. The length or bound on the length of two of the edges is also indicated in the figure.

- the fact that the distance between $\bar{p}$ and $\bar{q}$, is at least $|\lambda - \lambda'| \min \mathrm{alt}(\sigma)/\sqrt{n}$, thanks to Lemma 5.12 of [7]
- Lemma 26 to bound $\min \mathrm{alt}(\sigma)$
- Equation (34), which gives a bound on the angle $\angle \bar{p}x\bar{q}$, namely ϕ_0
- Lemma 27, gives that

$$\angle(\mathrm{aff}(\sigma^n)^\perp, N_p\mathcal{M}) \leq \arcsin\left(\frac{(\alpha_{d-n})^{4+n}}{6(n+1)}\zeta^n\right) \leq \arcsin\left(\frac{(\alpha_{d-n})^3}{3}\right), \tag{35}$$

where $\mathrm{aff}(\sigma^n)^\perp$ denotes the space perpendicular to $\mathrm{aff}(\sigma^n)$. Because $e_j \in N_p\mathcal{M}$, combining this with Lemma 23 and the triangle inequality for angles yields

$$\begin{aligned}\angle(N_{\bar{p}'}(e_j), \mathrm{aff}(\sigma^n)^\perp) &\leq \angle(N_{\bar{p}'}(e_j), e_j) + \angle(N_p\mathcal{M}, \mathrm{aff}(\sigma^n)^\perp) \\ &\leq \arccos\left(\frac{16}{10}\alpha_{d-n}\right) + \arcsin\left(\frac{(\alpha_{d-n})^3}{3}\right). \end{aligned}\tag{36}$$

We need a lower bound on $\sin(\angle \bar{p}\bar{q}x)$ and $\sin(\angle \bar{q}\bar{p}x)$, that is

$$\sin \angle(N_{\bar{p}'}(e_j), \mathrm{aff}(\sigma^n)) = \cos \angle(N_{\bar{p}'}(e_j), \mathrm{aff}(\sigma^n)^\perp).$$

We also remind ourselves of the trigonometric identity

$$\cos\left(\arccos(a) + \arcsin(b)\right) = a\sqrt{1-b^2} - b\sqrt{1-a^2}.$$

Using (36) now gives

$$\begin{aligned}\sin \angle(N_{\bar{p}'}(e_j), \mathrm{aff}(\sigma^n)) &\geq \cos\left(\arccos\left(\frac{16}{10}\alpha_{d-n}\right) + \arcsin\left(\frac{(\alpha_{d-n})^3}{3}\right)\right) \\ &= \left(\frac{16}{10}\alpha_{d-n}\right)\sqrt{1 - \frac{(\alpha_{d-n})^6}{3^2}} - \frac{(\alpha_{d-n})^3}{3}\sqrt{1 - \left(\frac{16}{10}\alpha_{d-n}\right)^2} \\ &\geq \frac{16}{10}\alpha_{d-n} - \left(\frac{16}{10}\alpha_{d-n}\right)\frac{(\alpha_{d-n})^6}{3^2} - \frac{(\alpha_{d-n})^3}{3} \\ &\geq \frac{16}{10}\alpha_{d-n} - \frac{5(\alpha_{d-n})^3}{9} \\ &\geq \alpha_{d-n}. \end{aligned}\tag{37}$$

The considerations we summed up yield:

$$
\begin{aligned}
|\bar{p}x|, |\bar{q}x| &\geq \frac{(|\lambda - \lambda'| \min \mathrm{alt}(\sigma)/\sqrt{n})\alpha_{d-n}}{\left(\frac{10}{16\,\alpha_{d-n}}\right)^2 \sqrt{n+1}|\lambda - \lambda'|} \\
&\geq \frac{\min \mathrm{alt}(\sigma)\alpha_{d-n}\left(\frac{16}{10}\alpha_{d-n}\right)^2}{n+1} \\
&\geq \frac{\alpha_{d-n}\left(\frac{16}{10}\alpha_{d-n}\right)^2}{n+1}(\zeta\alpha_{d-n-1})^n\tilde{L}
\end{aligned}
$$

Using (37) again yields that the distance from x to $\mathrm{aff}(\sigma^n)$ is bounded from below by

$$
d(x, \mathrm{aff}(\sigma^n)) \geq \frac{(\alpha_{d-n})^2\left(\frac{16}{10}\alpha_{d-n}\right)^2}{n+1}(\zeta\alpha_{d-n-1})^n\tilde{L}.
$$

◀

Proof of Lemma 29. Consider $v(\tau^d) \subset K \cap \tau^d$, where we use the definition (19) and choose an arbitrary n dimensional simplex $\sigma^n \subset K \cap \tau^d$. Note that $v(\tau^d) \in K \cap \tau^d$. Thanks to Lemma 27,

$$
\sin \angle(\mathrm{aff}(\sigma^n), T_v\mathcal{M}) \leq \frac{(\alpha_{d-n})^{4+n}}{6(n+1)}\zeta^n.
$$

From this bound we conclude that

$$
d_H(T_v\mathcal{M} \cap B(v, 2L), \mathrm{aff}(\sigma^n) \cap B(v, 2L)) \leq 2\frac{(\alpha_{d-n})^{4+n}}{6(n+1)}\zeta^n L,
$$

where d_H denotes the Hausdorff distance. Because of Lemma 6 and (8), we have that

$$
d_H(T_v\mathcal{M} \cap B(v, 2L), \pi_v^{-1}(B_{T_v\mathcal{M}}(v, 2L))) \leq \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n} L,
$$

where $B_{T_v\mathcal{M}}(v, 2L)$ denotes the ball in $T_v\mathcal{M}$ with radius $2L$ and centre v. This gives us

$$
\begin{aligned}
&d_H(\mathrm{aff}(\sigma^n) \cap B(v, 2L), \pi_v^{-1}(B_{T_v\mathcal{M}}(v, 2L))) \\
&\qquad \leq 2\frac{(\alpha_{d-n})^{4+n}}{6(n+1)}\zeta^n L + \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n} L \qquad \text{(by the triangle inequality)} \\
&\qquad \leq \frac{(\alpha_{d-n})^{4+n}\zeta^n}{n+1}L.
\end{aligned}
$$

Because $\mathcal{M} \cap \tau \subset \pi_v^{-1}(B_{T_v\mathcal{M}}(v, 2L))$ and the distance between $M \cap \tau$ and $\mathrm{aff}(\sigma^n)$ is small compared to the size of the neighbourhood of K given in Lemma 28, that is

$$
\frac{(\alpha_{d-n})^{4+n}\zeta^n}{n+1}L \leq \frac{\left(\frac{16}{10}\right)^2(\alpha_{d-n})^4}{n+1}\zeta^n(\alpha_{d-n-1})^n\tilde{L}, \tag{38}
$$

$\mathcal{M} \cap \tau$ is contained in this neighbourhood of K. ◀